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AN INTRODUCTION TO STEINER POLYNOMIALS OF GRAPHS

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ABSTRACT

In this paper, we introduce a new concept of Steiner polynomial of a connected graph G. The Steiner polynomial of G is the polynomial $S(G, x) = \sum_{i = s(G)}^{|V(G)|} s(G, i)\, x^i$ *, where s(G, i) is the number of Steiner sets of G of size i and s (G) is the Steiner number of G. We obtain some properties of S (G, x) and its coefficients. Also, we compute the polynomials for paths.*

Key words: *Steiner set, Steiner polynomial, Steiner number.*

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1. INTRODUCTION

For a connected graph G and a set $W \subseteq V(G)$, a tree contained in G is a Steiner tree with respect to W if T is a tree of minimum order with $W \subseteq V(G)$. The set S (W) contains, of all vertices in G that lie on some Steiner tree with respect to W. The minimum cardinality among the Steiner sets of G is the Steiner number, s (G). We denote the family of Steiner sets of a connected graph G with cardinality i by S (G, i).

Each extreme vertex of a graph G belongs to every Steiner set of G. In particular, each end-vertex of G belongs to every Steiner set of G.

Every non trivial tree with exactly k end- vertices has Steiner number k.

A graph in which any two distinct vertices are adjacent is called a complete graph. The complete graph with n vertices is denoted by K_n.

A graph G is called a bipartite graph if V (G) of G can be partitioned into two disjoint subsets V_1 and V_2 such that every edge G joins a vertex of V_1 to a vertex of V_2. If V_1 contains m vertices and V_2 contains n vertices then the complete bigraph G is denoted by $K_{m, n}$. $K_{1, m}$ is called a star for $m \geq 2$.

The complement of a complete graph K_n is denoted by $\overline{K}_n$ and it is a null graph.

If K_m and K_n are two complete graphs of order m and n respectively, then the graph $K_m \bigcup_{V_o} K_n$ is a graph of order $m + n - 1$ with a common cut vertex v_0.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \phi$. Then, the Sum $G_1 + G_2$ is the graph $G_1 \cup G_2$ together with all the edges joining the vertices of V_1 to the vertices of V_2.

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ie., If G_1 is a (p_1, q_1) graph and G_2 is (p_2, q_2) graph, then $G_1 + G_2$ is a $(p_1 + p_2, q_1 + q_2 + p_1 p_2)$ graph.

A walk is called a path if all its points are distinct. A path of order n is denoted by P_n.

A wheel, W_n, is a graph with n vertices $v_1, v_2 . v_n$ with v_1 having degree n – 1 and all the remaining (n - 1) vertices having degree 3, v_i is adjacent to v_{i+1} and v_n is adjacent to v_2.

The corona of two graphs G_1 and G_2, as defined by Frucht and Harary in [3] is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. The corona $G_1 \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $u \in V(G)$, a new vertex v' and a pendent edge vv' are added.

2. STEINER POLYNOMIAL OF A GRAPH

Definition 2.1: Let S (G, i) be the family of Steiner sets of a graph G with cardinality i and let s (G, i) = | S (G, i) | . Then the Steiner Polynomial, S (G, x) of G is defined as

$$S(G, x) = \sum_{i = s(G)}^{|V(G)|} s(G, i)\ x^i, \text{ where } s(G) \text{ is the Steiner number of } G.$$

Example 2.2: For the graph G, in Figure 1, let $W_1 = \{ v_1, v_4, v_6\}$. Then the trees T_1, T_2, T_3, T_4 given in Figure 2 are four distinct Steiner W_1- trees of order 5 such that every vertex of G lies on some Steiner W_1 - trees and so W_1 is a Steiner set of G.

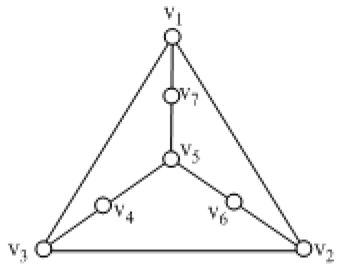


Figure: 1

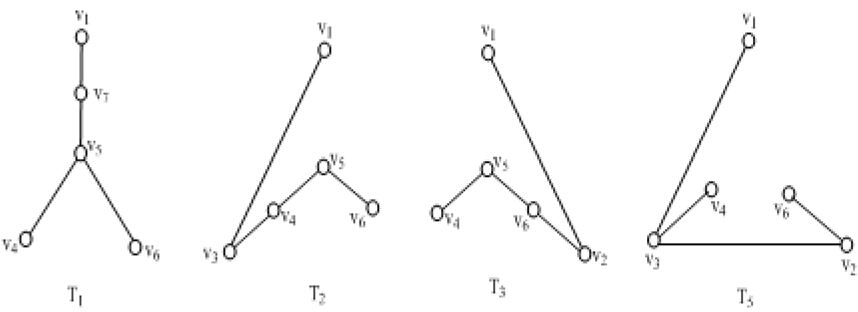


Figure: 2

Since there is no 2-element Steiner set of G, W_1 is a minimum Steiner set of G so that s (G) = 3.

The other Steiner sets with cardinality 3 are $W_2 = \{v_2, v_4, v_7\}$ and $W_3 = \{v_3, v_6, v_7\}$.

S (G, i) is the family of Steiner sets with cardinality i.
S (G, 3) = {{ v_1, v_4, v_6}, {v_2, v_4, v_7}, {v_3, v_6, v_7}}

Hence, s (G, 3) = | S (G, 3)| = 3

A Steiner set with cardinality 4 is $W_4 = \{v_1, v_2, v_3, v_5\}$. The Steiner W_4 trees are as follows:

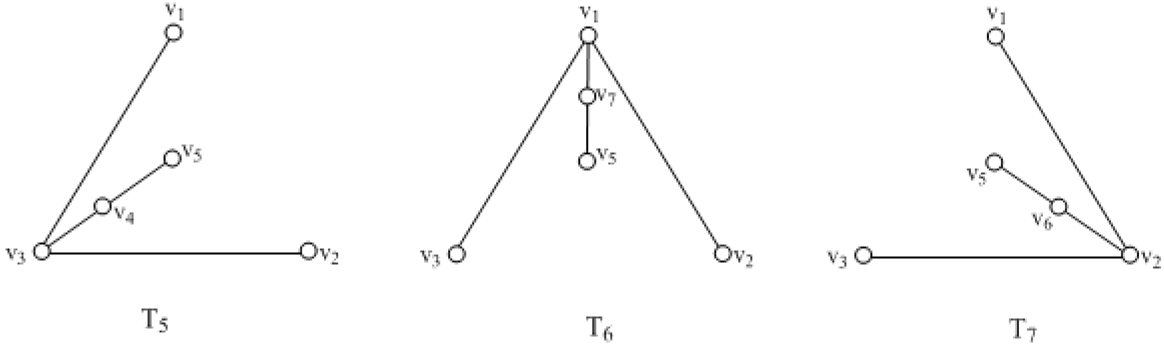


Figure: 3

The other Steiner sets with cardinality 4 are $\{v_1, v_2, v_4, v_5\}$, $\{v_1, v_3, v_5, v_6\}$ and $\{v_2, v_3, v_5, v_7\}$

$\therefore$ S (G, 4) = $\{\{v_1, v_2, v_3, v_5\}, \{v_1, v_2, v_3, v_5\}, \{v_1, v_3, v_4, v_5\}, \{v_2, v_3, v_5, v_7\}$

Hence, S (G, 4) = 4.

Also, s (G, 5) = $\{\{v_1, v_2, v_4, v_6, v_7\}, \{v_1, v_3, v_4, v_6, v_7\}, \{v_2, v_3, v_4, v_6, v_7\}\}$

Therefore, s (G, 5) = 3

There is no Steiner set with cardinality 6, because, if we take any six vertices out of 7 vertices, there is a tree of order 6. To include the 7th vertex a tree should have order 7 including the other 6 vertices

$\therefore$ S (G, 6) = { }

Therefore, s (G, 6) = 0.

The whole set $\{v_1, v_2, \ldots v_7\}$ is also a Steiner set.

ie, S (G, 7) = $\{\{v_1, v_2, \ldots V_7\}\}$

Therefore, s (G, 7) = 1

$$\text{Hence,}\quad S(G, x) = \sum_{i = s(G)}^{|V(G)|} s(G, i)\, x^i = 3x^3 + 4x^4 + 3x^5 + x^7$$

Theorem 2.3: If $G_1 \cong G_2$, then S (G_1, x) = S (G_2, x).

Proof: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the given isomorphic graphs.

Since $G_1 \cong G_2$, there exists a bijection f: $V_1 \rightarrow V_2$ such that v_i and v_j are end vertices/ extreme vertices in G_1 iff f (v_i) and f (v_j) are end vertices/ extreme vertices in G_2.

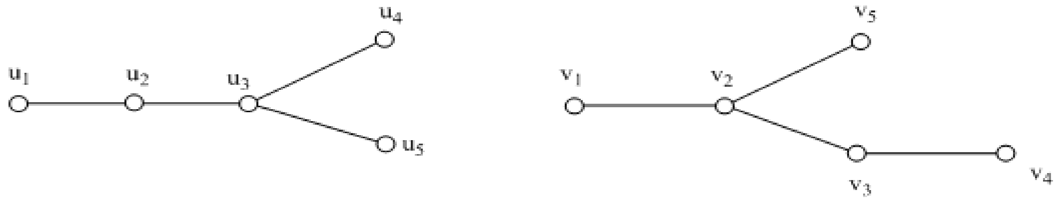
Hence, there is a one to one correspondence between the Steiner sets of G_1 and the Steiner sets of G_2.

Therefore, s (G_1, i) = s (G_2, i), $\forall$ i.

If S (G_1, x) and S (G_2, x) are the Steiner polynomials of G_1 and G_2 respectively, then S (G_1, x) = S (G_2, x).

Remark: 2.4 Converse is not true.

Example: 2.5 Consider the following two graphs G_1 and G_2.



G_1 **G_2**

Figure: 4

Steiner sets of G_1 are

$\{u_1, u_4, u_5\}$

$\{u_1, u_2, u_4, u_5\}, \{u_1, u_3, u_4, u_5\}$

$\{u_1, u_2, u_3, u_4, u_5\}$

$\therefore$ S $(G_1, x) = x^3 + 2x^4 + x^5$ (1)

Steiner sets of G_2 are
$\{v_1, v_4, v_5\}$
$\{v_1, v_2, v_4, v_5\}$, $\{v_1, v_3, v_4, v_5\}$
$\{v_1, v_2, v_3, v_4, v_5\}$

$\therefore \quad S(G_2, x) = x^3 + 2x^4 + x^5$ (2)

From (1) and (2)

$S(G_1, x) = S(G_2, x)$

But, G_1 and G_2 are not isomorphic graphs.

Theorem 2.6: The Steiner polynomial of a complete bipartite graph $K_{m,n}$ is
$s(K_{m,n}, x) = x^n + x^m + x^{m+n}$; m, n > 1

Proof: Let $K_{m,n}$ be a complete bipartite graph with two partite sets X and Y so that | X | = m and | Y | = n.
Let $X = \{u_1, u_2, \ldots u_m\}$ and $Y = \{v_1, v_2, \ldots v_n\}$.

Without loss of generality, we assume m > n.

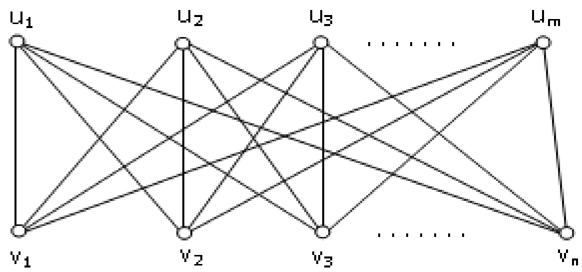


Figure 5

There are only three Steiner sets. Since n < m, the unique Steiner set with minimum cardinality n is Y.

$\therefore s(K_{m,n}, n) = 1$

The unique Steiner set with cardinality m is X.

$\therefore s(K_{m,n}, m) = 1$

The Steiner set with cardinality m + n is $X \cup Y$.

$\therefore s(K_{m,n}\ m+1) = 1$

There is no other Steiner sets for $K_{m,n}$. For, if $W = X \cup \{u_1\}$, then there is only one tree of order m +1 containing the elements of W. In this tree, only the elements of W are involved, but no other vertex of $K_{m,n}$ is involved. The other tree which contains the elements of W and the remaining vertices of $K_{m,n}$ is of minimum order m + 2.

$\therefore$ W is not a Steiner set.

$W_1 = X \cup \{v_i, v_j\}$, $i \neq j$, $1 \leq i$, $j \leq n$ is not a Steiner set.

Also, $W_2 = Y \cup \{u_j\}$, i = 1, 2, ... n is not a Steiner set.

Hence,

$$S(K_{m,n}, x) = \sum_{i = s(K_{m,n})}^{|V(K_{m,n})|} S(K_{m,n}, i)\, x^i$$
$$= x^n + x^m + x^{m+n}$$

Corollary 2.7: $S(K_{n,\,n}, x) = x^n (2 + x^n)$

Proof: Replace m by n in Theorem 2.6, we have

$$S(K_{m,\,n}, x) = x^n + x^n + x^{n+n}$$
$$= x^n (2 + x^n)$$

Theorem 2.8: $S(K_{1,\,n}, x) = x^n (1 + x)$

Proof: Let $V(K_{1,\,n})$ be $\{u, v_1, v_2, \ldots, v_n\}$

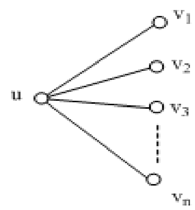


Figure 6

Since $v_1, v_2 \ldots, v_n$ are the end vertices, the minimum Steiner set is $\{v_1, v_2, \ldots, v_n\}$.

It is the unique minimum Steiner set.

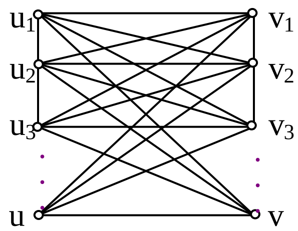
$\therefore \quad s(K_{1,\,n}, n) = 1$

The other Steiner set is $\{u, v_1, v_2, \ldots, v_n\}$

$$\therefore \quad S(K_{1,\,n}, x) = x^n + x^{n+1} = x^n (1 + x)$$

Theorem 2.9: Let G_1 and G_2 be any two connected graphs of order m and n respectively. Then

$S(G_1 + G_2, x) = x^{m+n}$



$G_1 + G_2$

Figure 7

Proof: If G_1 and G_2 are connected graphs of order m and n respectively, then $G_1 + G_2$ is also a connected graph of order m + n.

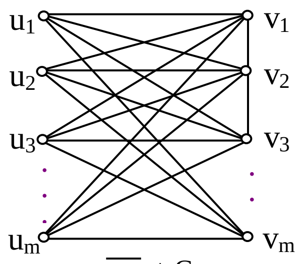
The unique Steiner set of $G_1 + G_2$ is $\{u_1, u_2, \ldots u_m, v_1, v_2, \ldots v_n\}$ of cardinality m + n.

$\therefore \; S(G_1 + G_2, x) = x^{m+n}$

Hence the proof.

Theorem 2.10: Let G be a connected graph of order n. Then

$$S(\overline{K_m} + G, x) = x^m (1 + x^n)$$



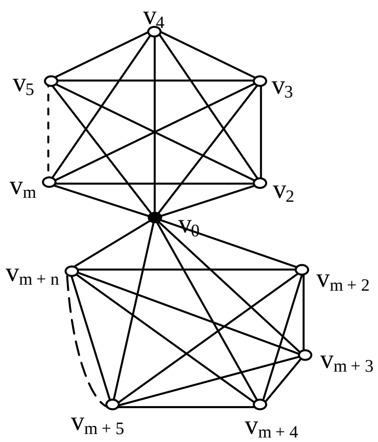
$\overline{K_m} + G_2$

Figure 8

Proof: There are only two Steiner sets for $\overline{K_m} + G$. They are $\{u_1, u_2, \ldots u_m\}$ of cardinality m and $\{u_1, u_2, \ldots u_m, v_1, v_2, \ldots v_n\}$ of cardinality m + n.

$$\therefore \; S(\overline{K_m} + G, x) = x^m + x^{m+n}$$
$$= x^m (1 + x^n)$$

Theorem 2.11: $S(K_m \bigcup_{v_0} K_n, x) = x^{m+n-2} (1 + x)$



$K_m \bigcup_{v_0} K_n$

Figure 9

Proof: Let $V(K_m) = \{v_0, v_2, v_3 \ldots v_m\}$ and $V(K_n) = \{v_0, v_{m+2}, v_{m+3}, \ldots v_{m+n}\}$

Since, every vertex of a complete graph is an extreme vertex, $s(K_m) = m$.

Since, v_0 is the cut vertex of $K_m \bigcup_{v_0} K_n$, the minimum Steiner set is $\{v_2, v_3, \ldots v_m, v_{m+2}, v_{m+3}, \ldots v_{m+n}\}$ of cardinality m + n – 2.

The other Steiner set is $\{v_0, v_2, v_3, ...v_m, v_{m+2}, V_{m+3} ... V_{m+n}\}$ of cardinality m + n – 1.

$$\therefore \; S(K_m \bigcup_{V_0} K_n, x) = x^{m+n-2} + x^{m+n-1}$$

$$= x^{m+n-2}(1 + x)$$

3. STEINER POLYNOMIAL OF G o K_1

Let G be any connected graph with vertex set $\{v_1, v_2, . . .v_n\}$. Add n new vertices $\{u_1, u_2 . . . u_n\}$ and join u_i to v_i for $1 \le i \le n$, by the definition of corona of two graphs. We shall denote this graph by G **o** K_1. In this section, we calculate the polynomial, S (G **o**K_1, x). Also, we show that s (G **o** K_1, x) is unimodal.

Lemma 3.1: For any connected graph G of order n, s (G **o** K_1, x) = n.

Proof: Since, every end vertex of the graph G **o** K_1 is an element of Steiner sets of it, the minimum Steiner set is the set of all its end vertices.

ie, W = $\{u_1, u_2,...u_n\}$ is the minimum Steiner set.

$\therefore$ s (G **o** K_1) = n.

By Lemma 3.1, s (G **o** K_1, m) = 0 for $m < n$, we calculate s (G **o** K_1, m) for $n \le m \le 2n$.

Theorem 3.2: For any graph G of order n and for $n \le m \le 2n$, s (G **o** K_1 , m) = $\binom{n}{m-n}$.

Hence, S (G **o** K_1, x) = $x^n (1 + x)^n$.

Proof:

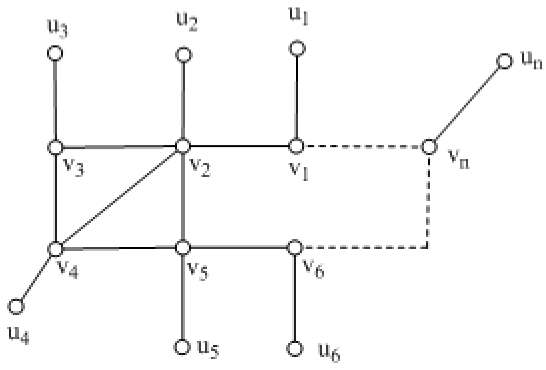


Figure 10

Suppose that W is a Steiner set of G **o** K_1 of cardinality m.

When m = n, the Steiner set with cardinality n is W = $\{u_1, u_3, ...u_n\}$.

$$\therefore \; s(G \circ K_1, n) = 1 = \binom{n}{0} = \binom{n}{m-n}$$

When m = n + 1, the Steiner sets with cardinality n + 1 are

$W_i = \{u_1, u_2,...u_n\} \cup \{v_i\}$ i = 1,2,...n.

$\therefore$ $s(G \circ K_1, n+1) = \binom{n}{1} = \binom{n}{m-n}$

When m = n + 2, the Steiner sets with cardinality n + 2 are

$W_l = \{u_1, u_2, \ldots u_n\} \cup \{v_i, v_j\}$, $1 \le i, j \le n$, $i \ne j$

$\therefore$ $s(G \circ K_1, n+2) = \binom{n}{2} = \binom{n}{m-n}$

Continuing this way, the Steiner set with cardinality m = 2n is the whole set

$\{u_1, u_2, \ldots u_n\} \cup \{v_1, v_2, \ldots v_n\}$

$\therefore$ $s(G \circ K_1, 2n) = 1 = \binom{n}{n} = \binom{n}{m-n}$

In general, we conclude that

$\therefore$ $S(G \circ K_1, m) = \binom{n}{m-n}$

$\therefore$ The Steiner polynomial of $G \circ K_1$ is

$$\begin{aligned} S(G \circ K_1, x) &= nC_0\, x^n + nC_1\, x^{n+1} + \ldots + nC_n\, x^{2n} \\ &= x^n (1 + nC_1 x + nC_2 x^2 + \ldots + nC_n x^n) \\ &= x^n (1+x)^n \end{aligned}$$

Here we discuss about unimodality of the Steiner Polynomial of $G_n \circ K_1$, where G_n denotes a graph with n vertices.

Let us denote $G_n \circ K_1$ by G_n^*.

Theorem 3.3: For every $n \in \mathbb{N}$,

$s(G_n^*, n) = s(G_n^*, 2n) = 1$.

Proof: By theorem 3.2, $s(G_n^*, n) = nC_0 = 1$ and $s(G_n^*, 2n) = nC_n = 1$.
Hence the theorem.

Theorem 3.4 (Unimodal theorem for $G \circ K_1$): For every $n \in \mathbb{N}$

(i) $1 = s(G^*_{3n}, 3n) < s(G^*_{3n}, 3n+1) < \ldots < s(G^*_{3n}, 4n-1) < s(G^*_{3n}, 4n) > \ldots > s(G^*_{3n}, 6n-1) > s(G^*_{3n}, 6n) = 1$

(ii) $1 = s(G^*_{3n+1}, 3n+1) < s(G^*_{3n+1}, 3n+2) < \ldots < s(G^*_{3n+1}, 4n) < s(G^*_{3n+1}, 4n+1) > s(G^*_{3n+1}, 4n+2) > \ldots > s(G^*_{3n+1}, 6n+1) > s(G^*_{3n+1}, 6n+2) = 1$

(iii) $1 = s(G^*_{3n+2}, 3n+2) < s(G^*_{3n+2}, 3n+3) < \ldots < s(G^*_{3n+2}, 4n+2) < s(G_{3n+2}, 4n+3) > s(G^*_{3n+2}, 4n+4) > \ldots > s(G^*_{3n+2}, 6n+3) > s(G^*_{3n+2}, 6n+4) = 1$

Proof:

(i) Obviously $s(G^*_{3n}, 3n) = 1$ and $s(G^*_{3n}, 6n) = 1$.
We shall prove that $s(G^*_{3n}, i) < s(G^*_{3n}, i+1)$ for $3n \le i \le 4n-1$ and $s(G^*_{3n}, i) > s(G^*_{3n}\ i+1)$ for $4n \le i \le 6n-1$.

Suppose that $s((G^*_{3n}, i) < s(G^*_{3n}, i + 1)$, by theorem 3.2, we have

$$\binom{3n}{i - 3n} < \binom{3n}{i - 3n + 1}$$

$\Rightarrow$ $i < 4n - 1$. But $i \geq 3n$

Hence $3n \leq i < 4n - 1$.

Similarly, we have $s(G^*_{3n}, i) > s(G^*_{3n}, i + 1)$ for $4n \leq i \leq 6n - 1$

Proof of parts (ii) and (iii) are similar as part (i).

4. STEINER SETS OF PATHS

Let P_n, $n \exists 2$ be a path with n vertices $V(P_n) = \{1, 2, \ldots n\}$ and $E(P_n) = \{\{1, 2\}, \{2, 3\}, \ldots \{n - 1, n\}\}$.

Let $S(P_n, i)$ be the family of Steiner sets of P_n with cardinality i. We investigate the Steiner sets of the path P_n.

Lemma 4.1: The following properties hold for paths:
(i) $s(P_n) = 2$, $n \geq 2$
(ii) $S(P_m, i) = \phi$ iff $i > m$ or $i < 2$

Proof:
(i) In a path P_n, there are two end vertices. The path P_n is the unique Steiner tree. Hence the minimum Steiner set has 2 elements.
$\therefore$ $s(P_n) = 2$
(ii) If follows from part (i) and the definition of Steiner set.

5. STEINER POLYNOMIALS OF PATHS

In this section, we introduce and investigate the Steiner polynomials of paths.

Let $S(P_n, i)$ be the family of Steiner sets of a path P_n with cardinality i and let $s(P_n, i) = |S(P_n, i)|$. Then the Steiner polynomial, $S(P_n, x)$ of P_n is

$$S(P_n, x) = \sum_{i=2}^{n} s(P_n, i)\, x^i.$$

Theorem 5.1: Let $S(P_n, i)$ be the family of Steiner sets of P_n with cardinality i.

Then (i) $|S(P_n, i)| = |S(P_{n-1}, i - 1)| + |S(P_{n-1}, i)|$
(ii) $S(P_n, x) = x\, S(P_{n-1}, x) + S(P_{n-1}, x)$
(iii) For every $n \geq 2$, $S(P_n, x) = x^2 (1 + x)^{n-2}$

Proof: Let $V(P_n) = \{v_1, v_2, \ldots v_n\}$
Every Steiner set of P_n contains the end vertices v_1 and v_n.

In this case the entire path is the Steiner tree.

If we fix v_1 and v_n, we have to choose any $i - 2$ vertices from the remaining $n - 2$ vertices of P_n, in order to get the Steiner sets of cardinality i.

$\therefore$ Here, we have $(n - 2)\, C_{i-2}$ Steiner sets of cardinality i.

$\therefore |S(P_n, i)| = (n - 2)\, C_{i-2}$

$|S(P_{n-1}, i - 1)| = (n - 3)\, C_{i-3}$ and $|S(P_{n-1}, i)| = (n - 3)\, C_{i-2}$

But, $(n-2)\, C_{i-2} = (n-3)\, C_{i-3} + (n-3)\, C_{i-2}$

Therefore, $|S(P_n, i)| = |S(P_{n-1}, i-1)| + |S(P_{n-1}, i)|$

(ii) By (i), we have

$$|S(P_n, i)| = |S(P_{n-1}, i-1)| + |S(P_{n-1}, i)|$$

When i = 2,

$$|S(P_n, 2)| = |S(P_{n-1}, 1)| + |S(P_{n-1}, 2)|$$
$$\Rightarrow \quad x^2\, |S(P_n, 2)| = x^2\, |S(P_{n-1}, 1)| + x^2\, |S(P_{n-1}, 2)|$$

when i = 3,

$$|S(P_n, 3)| = |S(P_{n-1}, 2)| + |S(P_{n-1}, 3)|$$
$$\Rightarrow x^3\, |S(P_n, 3)| = x^3\, |S(P_{n-1}, 2)| + x^3\, |S(P_{n-1}, 3)|$$

When i = 4,

$$|S(P_n, 4)| = |S(P_{n-1}, 3)| + |S(P_{n-1}, 4)|$$
$$\Rightarrow x^4\, |S(P_n, 4)| = x^4\, |S(P_{n-1}, 3)| + x^4\, |S(P_{n-1}, 4)|$$

.

.

When i = n – 1,

$$|S(P_n, n-1)| = |S(P_{n-1}, n-2)| + |S(P_{n-1}, n-1)|$$
$$\Rightarrow x^{n-1}\, |S(P_{n-1}, n-1)| = x^{n-1}\, |S(P_{n-1}, n-2)| + x^{n-1}\, |S(P_{n-1}, n-1)|$$

When i = n

$$|S(P_n, n)| = |S(P_{n-1}, n-1)| + |S(P_{n-1}, n)|$$
$$\Rightarrow \quad x^n\, |S(P_n, n) = x^n\, |S(P_{n-1}, n-1)| + x^n\, |S(P_{n-1}, n)|$$

Hence,

$$x^2|S(P_n, 2)| + x^3|S(P_n, 3)| + x^4|S(P_n, 4)| + \ldots + x^{n-1}|S(P_n, n-1)| + x^n|S(P_n, n)|$$
$$= [x^2|S(P_{n-1}, 1)| + x^2|S(P_{n-1}, 2)| + x^4\,|S(P_{n-1}, 3)| + \ldots + x^{n-1}|S(P_{n-1}, n-2)| + x^n|S(P_{n-1}, n-1)|] + [x^2|S(P_{n-1}, 2)| + x^3|S(P_{n-1}, 3)| + \ldots + x^{n-1}\,|S(P_{n-1}, n-1)| + x^n\,|S(P_{n-1}, n)|]$$

$$= x\,[x^2|S(P_{n-1}, 2)| + x^3|S(P_{n-1}, 3)| + \ldots + x^{n-2}|S(P_{n-1}, n-2)| + x^{n-1}|S(P_{n-1}, n-1)|] + [x^2|S(P_{n-1}, 2)| + x^3|S(P_{n-1}, 3)| + \ldots + x^{n-1}|S(P_{n-1}, n-1)|]$$
$$[\,|S(P_{n-1}, 1)| = |S(P_{n-1}, n)| = 0]$$

$$\sum_{i=2}^{n} |S(P_n, i)|\, x^i = x \sum_{i=2}^{n-1} |S(P_{n-1}, i)|\, x^i + \sum_{i=2}^{n-1} |S(P_{n-1}, i)|\, x^i$$

ie, $$\sum_{i=2}^{n} s(P_n, i)\, x^i = x \sum_{i=2}^{n-1} s(P_{n-1}, i)\, x^i + \sum_{i=2}^{n-1} s(P_{n-1}, i)\, x^i$$

ie, $S(P_n, x) = x\, S(P_{n-1}, x) + S(P_{n-1}, x)$

(iii) We prove this by induction on n.

When n = 2

$S(P_2, x) = x^2$

∴ The result is true for n = 2

Assume that the result is true for all natural numbers less than n.

ie, $S(P_{n-1}, x) = x^2 (1 + x)^{n-3}$

Now we prove the result for n

$$\begin{aligned} S(P_n, x) &= x\, S(P_{n-1}, x) + S(P_{n-1}, x) \\ &= x\,[x^2 (1 + x)^{n-3}] + x^2 (1 + x)^{n-3}] \\ &= x^2 (1 + x)^{n-3} (x + 1) \\ &= x^2 (1 + x)^{n-2} \end{aligned}$$

∴ The result is true for all n.

Using theorem 5.1, we get $s(P_n, i)$ for $2 \le n \le 15$ as shown in the Table 2.

Table 2: $s(P_n, i)$ is the number of Steiner sets of P_n with cardinality i.

n \ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	1													
3	0	1	1												
4	0	1	2	1											
5	0	1	3	3	1										
6	0	1	4	6	4	1									
7	0	1	5	10	10	5	1								
8	0	1	6	15	20	15	6	1							
9	0	1	7	21	35	35	21	7	1						
10	0	1	8	28	56	70	56	28	8	1					
11	0	1	9	36	84	126	126	84	36	9	1				
12	0	1	10	45	120	210	252	210	120	45	10	1			
13	0	1	11	55	165	330	462	462	330	165	55	11	1		
14	0	1	12	66	220	495	702	924	792	495	220	66	12	1	
15	0	1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1

Theorem 5.2: The following properties for the coefficients of $S(P_n, x)$ hold:

(i) $s(P_n, 2) = 1, \quad \forall\, n \ge 2$

(ii) $s(P_n, n) = 1, \quad \forall\, n \ge 2$

(iii) $s(P_n, n-1) = n - 2, \quad \forall\, n \ge 3$

(iv) $s(P_n, n-2) = \dfrac{(n-2)(n-3)}{2}, \quad \forall\, n \ge 4$

9v) $s(P_n, n-3) = \dfrac{(n-2)(n-3)(n-4)}{6}, \quad \forall\, n \ge 5$

(vi) $s(P_n, n-4) = \dfrac{(n-2)(n-3)(n-4)(n-5)}{24}, \quad \forall\, n \ge 6$

(vii) $s(P_n, i) = s(P_n, n - i + 2), \quad \forall\, n \ge 2$

(viii) If $S_n = \sum_{i=2}^{n} s(P_n, i)$, then, for every $n \ge 3$,
$S_n = 2(S_{n-1})$ with initial value $S_2 = 1$.

(ix) S_n = Total number of Steiner sets in $P_n = 2^{n-2}$.

Proof:

(i) There is a unique Steiner set contains the end vertices of cardinality two in P_n.

∴ $s(P_n, 2) = 1$, for all $n \ge 2$

(ii) The whole vertex set {[n]} is also a Steiner set.

∴ $s(P_n, n) = 1$, for all $n \ge 2$

(iii) We prove by induction on n.

The result is true for n = 3, since $s(P_3, 2) = 1$

Assume that the result is true for all natural numbers less than n.

Now, we prove it for n.

By theorem 5.1 (i) and part (ii), we have,

$$\begin{aligned} s(P_n, n-1) &= s(P_{n-1}, n-2) + s(P_{n-1}, n-1) \\ &= n - 3 + 1 \\ &= n - 2. \end{aligned}$$

$\therefore$ The result is true for all n.

(iv)We prove by induction on n.

The result is true for n = 4, since $s(P_4, 2). = 1$.

Assume that the result is true for all natural numbers less than n. Now, we prove it for n. By theorem 5.1 (i) and part (iii), we have

$$\begin{aligned} s(P_n, n-2) &= s(P_{n-1}, n-3) + s(P_{n-1}, n-2) \\ &= \frac{(n-3)(n-4)}{2} + (n-3) \\ &= \frac{(n-3)(n-4) + 2(n-3)}{2} \\ &= \frac{(n-3)(n-4+2)}{2} \\ &= \frac{(n-2)(n-3)}{2} \end{aligned}$$

$\therefore$ The result is true for all n.

(v) By induction on n.

The result is true for n = 5, since $s(P_5, 2) = 1$.

Assume that the result is true for all natural numbers less than n.

Now we prove it for n.

By theorem 5.1 (i) and part (iv), we have

$$\begin{aligned} s(P_n, n-3) &= s(P_{n-1}, n-4) + s(P_{n-1}, n-3) \\ &= \frac{(n-3)(n-4)(n-5)}{6} + \frac{(n-3)(n-4)}{2} \\ &= \frac{(n-3)(n-4)(n-5+3)}{6} \\ &= \frac{(n-2)(n-3)(n-4)}{6} \end{aligned}$$

$\therefore$ The result is true for all n.

(vi) By induction on n.

The result is true for n = 6, since $s(P_6, 2) = 1$

Assume that the result is true for all natural numberless than n.

Now, we prove it for n.

By theorem 5.1 (i) and part (v), we have

$$\begin{aligned} s(P_n, n-4) &= s(P_{n-1}, n-5) + s(P_{n-1}, n-4) \\ &= \frac{(n-3)(n-4)(n-5)(n-6)}{24} + \frac{(n-3)(n-4)(n-5)}{6} \\ &= \frac{(n-3)(n-4)(n-5)(n-6+4)}{24} \\ &= \frac{(n-2)(n-3)(n-4)(n-5)}{24} \end{aligned}$$

∴ The result is true for all n.

(vii) By induction on n

The result is true for n = 3, since $s(P_3, 2) = s(P_3, 3) = 1$

Assume that the result is true for all natural number less than n.

We now prove it for n.

By theorem 5.1 (i), we have

$$\begin{aligned} s(P_n, i) &= s(P_{n-1}, i-1) + s(P_{n-1}, i) \\ &= s(P_{n-1}, (n-1)-(i-1)+2) + s(P_{n-1}, (n-1)-i+2) \\ &= s(P_{n-1}, n-i+2) + s(P_{n-1}) n-i+1) \\ &= s(P_n, n-i+2) \end{aligned}$$

∴ The result is true for all n.

(viii) $S_n = \sum_{i=2}^{n} s(P_n, i)$

By theorem 5.1 (i), we have

$$\begin{aligned} S_n &= \sum_{i=2}^{n} [s(P_{n-1}, i-1) + S(P_{n-1}, i) \\ &= \sum_{i=2}^{n-1} s(P_{n-1}, i) + \sum_{i=2}^{n-1} s(P_{n-1}, i) \\ &= S_{n-1} + S_{n-1} \\ S_n &= 2\, S_{n-1}. \end{aligned}$$

(ix) By induction on n

When n = 3,

$$S_3 = 2 = 2^1 = 2^{3-2}$$

∴ The result is true for n = 3

Assume that the result is true for all natural numbers less than n.

∴ $S_{n-1} = 2^{n-3}$

Now, $$\begin{aligned} S_n &= 2\, S_{n-1} \\ &= 2 \times 2^{n-3} \\ &= 2^{n-2} \end{aligned}$$

$\therefore$ The result is true for all n

Hence the theorem.

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