

ON h – CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT

In this paper, we introduce a new class of sets namely, h – closed sets and their properties, Applying these sets, we introduce new spaces namely ${}_aT_h$, ${}_cT_h$ and sT_h spaces.

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1. INTRODUCTION

N. Levine [6] introduced the class of g – closed sets in 1970. M. K. R. S. Veera Kumar introduced several generalised closed sets namely, g^* - closed sets, *g - closed sets, α^*g - closed sets, *g_s - closed sets and ω – closed sets. In this paper, we introduce h – closed sets and applying these sets, we introduce new spaces namely ${}_aT_h$, ${}_cT_h$ and sT_h spaces.

2. PRELIMINARIES

Throughout this paper (X, τ) , (Y, σ) and (Z, η) represent topological spaces on which no separation axioms unless otherwise mentioned. For a subset A of a space (X, τ) , $cl(A)$, $int(A)$ and A^c denote the closure of A , the interior of A and the complement of A in X respectively.

2.1 Definition: A subset A of a space (X, τ) is called

1. a semi-open set [7] if $A \subseteq cl(int(A))$ and a semi closed set if $int(cl(A)) \subseteq A$
2. a preopen set [8] if $A \subseteq int(cl(A))$ and a preclosed set if $cl(int(A)) \subseteq A$
3. a α -open set [9] if $A \subseteq int(cl(int(A)))$ and a α - closed set if $cl(int(cl(A))) \subseteq A$

The intersection of all semi-closed (resp., preclosed, α -closed)-sets containing A of X is called the semi closure (resp., preclosure, α -closure) of A and is denoted by $scl(A)$ (resp, $pcl(A)$, $\alpha cl(A)$). The union of all semi - open sets contained in A is called semi interior of A and it is denoted by $sint(A)$.

2.2. Definition:

A subset A of a space (X, τ) is called a

1. a generalized closed (briefly g -closed) set [6] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) ; the compliment of a g - closed set is called a g -open set.
2. a generalized semi-closed (briefly gs -closed) set [3] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
3. a generalized semi pre-closed (briefly gsp -closed) set [5] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
4. a semi- generalized closed (briefly sg -closed) set [4] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ) ; the compliment of a sg - closed set is called a sg -open set.
5. a ψ - closed set [10] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg -open set in (X, τ) ;
6. a g^* -closed set [12] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open set in (X, τ) .

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7. a ω - closed set [11] if $cl(A) \subset U$ whenever $A \subset U$ and U is semi -open set in (X, τ) ; the compliment of a ω - closed set is called a ω -open set.
8. a g^* -closed set [12] if $cl(A) \subset U$ whenever $A \subset U$ and U is ω -open set in (X, τ) .
9. a g^* -semi closed (g^* s -closed) set [12] if $scl(A) \subset U$ whenever $A \subset U$ and U is g -open set in (X, τ) .
10. a α^* - g -closed set [12] if $acl(A) \subset U$ whenever $A \subset U$ and U is ω -open set in (X, τ) .

2.3 Definition:

1. a $T_{1/2}$ space [4] if every g - closed set is closed.
2. a $T_{1/2}^*$ space [13] if every g^* - closed set is closed.
3. a $T_{1/2}^*$ space [13] if every g -closed set is g^* - closed.

3. BASIC PROPERTIES OF h – CLOSED SETS

3.1 Definition: A subset A of (X, τ) is called a h - closed set if $scl(A) \subset U$ whenever $A \subset U$ and U is ω - open in X . The class of all h - closed subsets of (X, τ) is denoted by $hC(X, \tau)$.

3.2 Proposition: Every closed set (resp. α – closed set, semi – closed set) is h -closed. But the converses are not true as can be seen from the following examples.

Proof: Follows from the definition.

3.3 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. $hC(X, \tau) = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\}$. The set $\{a, b\}$ is h -closed but not closed (resp. not α – closed set, not semi – closed)

3.4 Proposition: Let X be a topological space and $A \subset X$. Then the following are true.

1. If A is g^* - closed, then A is h - closed set.
2. If A is g^* - closed, then A is h - closed set.
3. If A is α^* - closed, then A is h - closed set.
4. If A is g^* s - closed, then A is h - closed set.
5. If A is ψ - closed, then A is h - closed set.

Proof: (1) Since every ω – open set is g – open and $scl(A) \subset cl(A)$ the proof follows.

(2), (3), and (4): Since $scl(A) \subset acl(A) \subset cl(A)$, the proof is clear.

(5): Since every ω – open set is sg – open, the proof is clear.

3.5 Remarks: The converse of Proposition 3.4 is not true. For,

3.6 Example Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}\}$. The set $\{a\}$ is h closed but it is not a g^* - closed set.

3.7 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. The set $\{a, b\}$ is h - closed but it is not g^* - closed also it is not α^* - closed.

3.8 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. The set $\{a, b\}$ is a h - closed but it is not g^* s- closed set. Also it is not ψ - closed.

3.9 Proposition: Let X be a topological space and $A \subset X$. Then the following are true.

1. If A is h - closed, then A is gs - closed set.
2. If A is h - closed, then A is gsp - closed set.

Proof: (1) Since every open set is ω – open, the proof is clear.

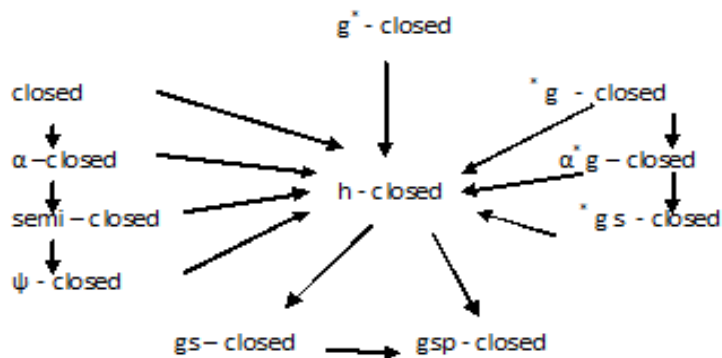
(2): Since every ω – open set is open and $spcl(A) \subset scl(A)$ every h - closed set is gs – closed.

3.10 Remarks: The converse of Proposition 3.4 is not true. For,

3.11 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. The set $\{a, b\}$ is gs- closed but not h-closed.

3.12 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a, b\}\}$. the set $\{a\}$ is gsp- closed but not h-closed.

The following diagram shows the relationships considered in this paper. ($A \longrightarrow B$ represents A implies B but B need not imply A .)



3.13 DEFINITION: Let (X, τ) be a topological space and $A \subset X$. The set of all limit points of A is said to be the Semi – derived set of A and is denoted by $D_s(A)$.

3.14 Theorem: Let A and B be two h – closed sets such that $D(A) \subset D_s(A)$ and $D(B) \subset D_s(B)$. Then $A \cup B$ is h – closed.

Proof: Let U be a ω – open set such that $A \cup B \subset U$. $A \subset U$ and U being ω – open $scl(A) \subset U$. $B \subset U$ and U being ω – open $scl(B) \subset U$. Therefore $scl(A) \cup scl(B) \subset U$. Given $D(A) \subset D_s(A)$ but always we have $D_s(A) \subset D(A)$. Therefore $D(A) = D_s(A)$. Hence $cl(A) = scl(A)$ and $cl(B) = scl(B)$. $scl(A \cup B) \subset U$. Therefore $A \cup B$ is h – closed.

3.15 Result: Intersection of two h – closed sets need not be a h – closed set.

3.16 Example: $X = \{a, b, c\}$ $\tau = \{\emptyset, X, \{a\}\}$. $\{a, b\}$ and $\{c, a\}$ are h – closed sets but $\{a, b\} \cap \{c, a\} = \{a\}$ is not a h – closed set.

3.17 Proposition: Let A and B be any two subsets of a space X .

- (1). If A is h – closed then $scl(A) - A$ contains no non empty ω - closed set.
- (2). If A is h – closed and $A \subseteq B \subseteq scl(A)$ then B is h – closed.

Proof: (1) Let F be a ω - closed subset of $scl(A) - A$. Then $F \subset scl(A)$ that is $(scl(A))^c \subset F^c$ and $F^c \supset A$. Since F is ω – closed F^c is ω – open and $A \subset F^c$. Hence $(scl(A)) \subset F^c$. Therefore $F \subset scl(A) \subset X - F = F^c$. This is possible only if $F = \emptyset$.

(2): Let U be a ω – open set such that $B \subseteq U$. Then $A \subseteq U$ given A is h – closed. Therefore $scl(A) \subseteq U$. Now $scl(B) \subseteq scl(scl(A)) = scl(A) \subseteq U$. Hence B is h – closed.

3.18 DEFINITION: A subset A of a topological space (X, τ) is said to be h – open if A^c is h – closed. The class of all h – open subsets of X is denoted by $hO(\tau)$.

3.19 Theorem: A subset A of a topological space (X, τ) is h – open if and only if $F \subseteq sint(A)$ whenever $A \supset F$ and F is ω – closed in X .

Proof: Suppose that is A h – open in X and $A \supset F$, where F is ω – closed in X . $A^c \subseteq F^c$ and F^c is ω – open in X . Hence $scl(A^c) \subseteq F^c$ implies $(sint(A))^c \subseteq F^c$. Thus we have $sint(A) \supset F$ that is $F \subseteq sint(A)$.

Converse

Suppose $A^c \subseteq U$ and U is ω – open. $A \supset U^c$ and U^c is ω – closed. Hence by the given hypothesis $\text{sint}(A) \supset U^c$. Therefore $(\text{sint}(A))^c \subseteq U$. Hence $\text{scl}(A^c) \subseteq U$.

3.20 Theorem: In a topological space (X, τ) for each $x \in X$ either $\{x\}$ is ω – closed or h – open in X .

Proof: Suppose $\{x\}$ is not ω – closed in X then $\{x\}^c$ is ω – closed and the only ω – open set containing $\{x\}^c$ is the space X itself. Therefore $\text{scl}(\{x\}^c) \subseteq X$ and so $\{x\}^c$ is h – closed– open.

4. APPLICATION OF h – CLOSED SETS

4.1 Definition: A space (X, τ) is called ${}_aT_h$ space if every h - closed set is a α closed set.

4.2 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. $hC(X, \tau) = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\} = {}_\alpha C(X, \tau)$. Thus (X, τ) is a ${}_aT_h$ space. The space in the following example is not a ${}_aT_h$ space.

4.3 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. $hC(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{b, c\}, \{c, a\}\}$. ${}_\alpha C(X, \tau) = \{\emptyset, X, \{c\}, \{b, c\}, \{a, c\}\}$.

Thus (X, τ) is not a ${}_aT_h$ space.

4.4 Theorem: If (X, τ) is a ${}_aT_h$ space, then every singleton of X is either ω – closed or α – open.

Proof: Let $x \in X$ and suppose that $\{x\}$ is not ω – closed. Then $X - \{x\}$ is not ω – open. This implies that X is the only ω - open set containing $X - \{x\}$. So $X - \{x\}$ is a h – closed set of (X, τ) . Since (X, τ) is an ${}_aT_h$ space, $X - \{x\}$ is α – closed or equivalently $\{x\}$ is α – open.

The converse of the above theorem is not true as it can be seen by the following example.

4.5 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. $\{a\}$ and $\{b\}$ are α – open sets of (X, τ) and $\{c\}$ is a ω – closed set of (X, τ) . But the space (X, τ) is not an ${}_aT_h$ space .

4.6 Definition: A space (X, τ) is called ${}_cT_h$ space if every h - closed set is a closed set.

4.7 Theorem: If (X, τ) is a ${}_cT_h$ space, then every singleton of X is either ω – closed or open.

Proof: Proof is similar to theorem 4.4

The converses of the above theorem is not true as it can be seen by the following examples.

4.8 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ $\{a\}$ and $\{b\}$ are open sets of (X, τ) and $\{c\}$ is a ω – closed set of (X, τ) . But the space (X, τ) is not a ${}_cT_h$ space.

4.9 Theorem: Every ${}_cT_h$ space is an ${}_aT_h$ space .

Proof: The Proof follows since every closed set is α – closed.

4.10 Theorem: ${}_cT_h$ is independent of $T_{1/2}$ ness and ${}^*T_{1/2}$ ness. It is shown by the following examples.

4.11 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$.

Then (X, τ) is a ${}_cT_h$ space but not a $T_{1/2}$ space and a ${}^*T_{1/2}$ space.

4.12 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.

Then (X, τ) is a $T_{1/2}$ space and a ${}^*T_{1/2}$ space but not a ${}_cT_h$ space.

4.13 Theorem: Every cT_h space is a $T_{1/2}^*$ space.

Proof: Let (X, τ) be a cT_h space. Let A be a g^* -closed set. Every g^* -closed set is h – closed.

Hence A is closed. Therefore (X, τ) is a $T_{1/2}^*$ space.

The converse of the above theorem is not true as it can be seen by the following example.

4.14 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. (X, τ) is a $T_{1/2}^*$ space but not a cT_h space.

4.15 Definition: A space (X, τ) is called a T_b space if every g_s -closed set is a closed set.

4.16 Theorem: Every T_b space is a sT_h space.

Proof: Since every h – closed set is g_s – closed set, the proof is clear.

The converse of the above theorem is not true as it can be seen by the following examples.

4.17 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. $hC(X, \tau) = \{\emptyset, X, \{a\}, \{b, c\}\} = C(X, \tau)$. $gsC(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\}$. The set $\{b\}$ is g_s – closed but not closed.

4.18 Definition: A space (X, τ) is called sT_h space if every h- closed set is a semi closed set.

4.19 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. (X, τ) is a sT_h space.

4.20 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. (X, τ) is not a sT_h space.

4.21 Theorem: If (X, τ) is a sT_h space then every singleton of X is either ω – closed or semi - open.

Proof: Proof is similar to theorem 4.4.

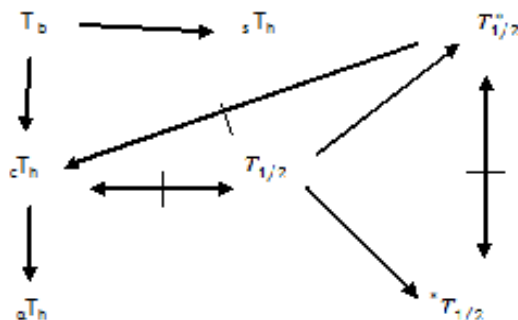
4.22 Theorem: Every T_b space is a sT_h space.

Proof: Since every h – closed set is g_s – closed set, the proof is clear.

The converse of the above theorem is not true as it can be seen by the following example.

4.23 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. $hC(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{b, c\}, \{c, a\}\} = sC(X, \tau) = gsC(X, \tau)$. The set $\{a\}$ is g_s – closed but not closed.

4.27 Remark: The following diagram shows some relationships considered in this paper. ($A \longleftrightarrow B$ represents A implies B but B need not imply A . $A \longrightarrow B$ means A and B are independent).



REFERENCES:

- [1] M.E. Abd El-Monsef, S.N. El-Deeb and R.A. Mahmoud, β – open sets and, B- continuous mappings. Bull. Fac . Sci Univ. (1983), 77 – 90.
- [2] D. Andrijevic, semi – preopen sets. Mat. Vesnik. (1986), 24 – 32.
- [3] S. P. Arya and T. Nour, characterization of s – normal spaces, Indian J . Pure. Appl, Math., (1990), 717 – 719.
- [4] P. Bhattacharya and B. K. Lahiri, Semi – generalized closed sets in topology, Indian J. Pure. Appl, Math., closed sets in topology, Indian J. Math. (1987), 375 – 382.
- [5] J. Dontchev, On generalizing semi – preopen sets. Mem. Fac. Sci. Kochi Uni . Ser. A Math., (1995), 35 – 48.
- [6] N. Levine. Generalized closed sets in topology, Rend. Circ. Math. Palermo, (1970), 89 – 96.
- [7] N. Levine, Semi – open sets and semi – continuity in topological spaces, Amer. Math. Monthly, (1963), 36 – 41.
- [8] A.S. Mashhour, M.E.Abd El-Monsef, S.N. El-Deeb, On pre – continuous and weak pre – continuous mappings, Proc. Math. And Phys. Soc. Egypt. (1982) 47 – 53.
- [9] O. Njastad, On some classes of nearly open sets, Pacific J. Math., (1965), 961 -970.
- [10] M.K.R.S. Veera Kumar, Between semi - closed sets and semi pre – closed sets, Rend. Istit. Mat. Univ. Trieste. XXXII (2000), 25 – 41.
- [11] M.K.R.S. Veera Kumar On \tilde{g} – closed sets in topological spaces, Bull Allahabad Math. Soc. (2003), 99 – 112.
- [12] M.K.R.S. Veera Kumar On μp – closed sets in topological spaces, Antarctica J. Math, 2(1) (2005), 31 – 52.
- [13] M.K.R.S. Veera Kumar, Between closed sets and g – closed sets Mem. Fac. Sci. Kochi Uni . Ser. A Math., (2000), 1 – 19.
