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COMMON FIXED POINT THEOREMS FOR FOUR MAPPINGS IN $\mathcal{M}$ - FUZZY METRIC SPACE

T. Veerapandi
Associate Professor of Mathematics, P.M.T. College, Melaneelithanallur – 627953, India
E-mail: tveerapandi@ymail.com

G. Uthaya Sankar*
Department of Mathematics, Mano College (An Institution of M.S.Univerity, Tirunelveli), Sankarankovil – 627756, India
E-mail: uthayaganapathy@yahoo.com

A. Subramanian
Department of Mathematics, The M.D.T. Hindu College, Tirunelveli – 627010, India
E-mail: asmani1963@gmail.com

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ABSTRACT

In this paper we prove a common fixed point theorem for four mappings in $\mathcal{M}$ – fuzzy metric space using the notion of semi compatibility. Also, we prove a common fixed point theorem for four weakly compatible mappings in $\mathcal{M}$ – fuzzy metric space.

***Mathematics Subject Classification**: 47H10, 54H25.*

***Keywords**: Complete $\mathcal{M}$ – Fuzzy metric space, Semi compatible mappings, weakly compatible mappings, Common fixed point.*

INTRODUCTION AND PRELIMINARIES

Zadeh [16] introduced the concept of fuzzy sets in 1965. George and Veeramani [2] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [7] and defined the Hausdorff topology of fuzzy metric spaces. Many authors [4, 8] have proved fixed point theorems in fuzzy metric space. Recently Sedghi and Shobe [13] introduced D^* - metric space as a probable modification of the definition of D - metric introduced by Dhage [1], and prove some basic properties in D^* - metric spaces. Using D^* - metric concepts, Sedghi and Shobe define $\mathcal{M}$ – fuzzy metric space and proved a common fixed point theorem in it. Jong Seo Park [5] introduced the concept of semi compatible and weak compatible in $\mathcal{M}$ – fuzzy metric space and prove some fixed point theorems satisfying some conditions in $\mathcal{M}$ – fuzzy metric space. In this paper we prove a common fixed point theorem for four mappings in $\mathcal{M}$ – fuzzy metric space using the notion of semi compatibility. Also, we prove a common fixed point theorem for four weakly compatible mappings in $\mathcal{M}$ – fuzzy metric space.

Definition: 1.1 Let X be a nonempty set. A generalized metric (or D^* - metric) on X is a function: $D^* : X^3 \to [0, \infty)$, that satisfies the following conditions for each $x, y, z, a \in X$
(i) $D^*(x, y, z) \geq 0$,
(ii) $D^*(x, y, z) = 0$ iff $x = y = z$,
(iii) $D^*(x, y, z) = D^*(p\{x, y, z\})$, (symmetry) where p is a permutation function,
(iv) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.
The pair (X, D^*), is called a generalized metric (or D^* - metric) space.

Example: 1.2 Examples of D^* - metric are
(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,
(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.
Here, d is the ordinary metric on X.

Corresponding author: G. Uthaya Sankar*, *E-mail: uthayaganapathy@yahoo.com

Definition: 1.3 A fuzzy set $\mathcal{M}$ in an arbitrary set X is a function with domain X and values in [0, 1].

Definition: 1.4 A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for all $a \in [0, 1]$,
(iv) $a*b \le c*d$ whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0, 1]$.

Examples for continuous t-norm are $a*b = ab$ and $a*b = \min\{a, b\}$.

Definition: 1.5 A 3-tuple $(X, \mathcal{M}, *)$ is called $\mathcal{M}$ – fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$

($FM-1$) $\mathcal{M}(x, y, z, t) > 0$
($FM-2$) $\mathcal{M}(x, y, z, t) = 1$ iff $x = y = z$
($FM-3$) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function
($FM-4$) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \le \mathcal{M}(x, y, z, t+s)$
($FM-5$) $\mathcal{M}(x, y, z, \cdot) : (0, \infty) \to [0, 1]$ is continuous
($FM-6$) $\lim_{t\to\infty} \mathcal{M}(x, y, z, t) = 1$.

Example: 1.6 Let X be a nonempty set and D^* is the D^* - metric on X. Denote $a*b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}$$

for all $x, y, z \in X$, then $(X, \mathcal{M}, *)$ is a $\mathcal{M}$- fuzzy metric space. We call this $\mathcal{M}$-fuzzy metric induced by D^* - metric space. Thus every D^* - metric induces a $\mathcal{M}$-fuzzy metric.

Lemma: 1.7 ([13]) Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. Then for every $t > 0$ and for every $x, y \in X$, we have $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

Lemma: 1.8 ([13]) Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ –fuzzy metric space. Then $\mathcal{M}(x, y, z, t)$ is non-decreasing with respect to t, for all x, y, z in X.

Definition: 1.9 Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ – fuzzy metric space and $\{x_n\}$ be a sequence in X

(a) $\{x_n\}$ is said to be converges to a point $x \in X$ if $\lim_{n\to\infty} \mathcal{M}(x, x, x_n, t) = 1$ for all $t > 0$
(b) $\{x_n\}$ is called Cauchy sequence if $\lim_{n\to\infty} \mathcal{M}(x_{n+p}, x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$
(c) A $\mathcal{M}$ – fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Remark: 1.10 Since $*$ is continuous, it follows from (FM-4) that the limit of the sequence is uniquely determined.

Definition: 1.11 Let S and T be two self mappings of a $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$. Then the mappings are said to be compatible if $\lim_{n\to\infty} \mathcal{M}(STx_n, TSx_n, TSx_n, t) = 1$, for all $t > 0$, whenever $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Definition: 1.12 Let S and T be two self mappings of a $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$. Then the mappings are called semi compatible if $\lim_{n\to\infty} \mathcal{M}(STx_n, Tz, Tz, t) = 1$, $\lim_{n\to\infty} \mathcal{M}(TSx_n, Sz, Sz, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Definition: 1.13 Let S and T be two self mappings of a $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$. Then the mappings S and T are said to be weakly compatible if they commute at their coincidence points; that is, if $Sx = Tx$ for some $x \in X$, then $STx = TSx$.

Lemma: 1.14 ([11]) Let $\{x_n\}$ be a sequence in a $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$ with the condition (FM-6). If there exists a number $k \in (0, 1)$ such that

$\mathcal{M}(x_n, x_{n+1}, x_{n+1}, kt) \ge \mathcal{M}(x_{n-1}, x_n, x_n, t)$ for all $t > 0$ and $n = 1, 2, 3 \ldots$, then $\{x_n\}$ is a Cauchy sequence.

Lemma 1.15 ([11]) Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ – fuzzy metric space with condition (FM-6). If there exists a number $k \in (0, 1)$ such that $\mathcal{M}(x, y, z, kt) \ge \mathcal{M}(x, y, z, t)$, for all $x, y, z \in X$ and $t > 0$, then $x = y = z$.

MAIN RESULTS:

Theorem: 2.1 Let S and T be two continuous self mappings of a complete $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$. Let A and B be two self mappings of X satisfying

(1) $A(X) \subset T(X)$, $B(X) \subset S(X)$.

(2) (A, S) and (B, T) are semi compatible.

(3) there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$\mathcal{M}(Ax, By, By, kt) \geq \min\{\mathcal{M}(By, Ty, Ty, t), \mathcal{M}(Sx, Ty, Ty, t), \mathcal{M}(Ax, Sx, Sx, t), \mathcal{M}(Ax, By, By, t)\}.$$

Then A, B, S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.

Since $A(X) \subset T(X)$, then there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$.

Also, since $B(X) \subset S(X)$, then there exists another point $x_2 \in X$ such that $Bx_1 = Sx_2$.

Then by induction, we can define a sequence $\{y_n\}$ in X such that

$y_{2n+1} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, \ldots$

Now using condition (3) we get

$$\begin{aligned}
\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt) &= \mathcal{M}(Ax_{2n}, Bx_{2n+1}, Bx_{2n+1}, kt) \\
&\geq \min\{\mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Ax_{2n}, Sx_{2n}, Sx_{2n}, t), \\
&\qquad \mathcal{M}(Ax_{2n}, Bx_{2n+1}, Bx_{2n+1}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+2}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n}, y_{2n}, t), \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+2}, t), \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n}, y_{2n}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t), \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t)\} \\
&= \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t)
\end{aligned}$$

Therefore $\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt) \geq \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t)$.

Also,
$$\begin{aligned}
\mathcal{M}(y_{2n+2}, y_{2n+3}, y_{2n+3}, kt) &= \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+3}, kt) \\
&= \mathcal{M}(y_{2n+3}, y_{2n+2}, y_{2n+2}, kt) = \mathcal{M}(Ax_{2n+2}, Bx_{2n+1}, Bx_{2n+1}, kt) \\
&\geq \min\{\mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Sx_{2n+2}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Ax_{2n+2}, Sx_{2n+2}, Sx_{2n+2}, t), \\
&\qquad \mathcal{M}(Ax_{2n+2}, Bx_{2n+1}, Bx_{2n+1}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+2}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+2}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+3}, y_{2n+2}, y_{2n+2}, t), \mathcal{M}(y_{2n+3}, y_{2n+2}, y_{2n+2}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+2}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+3}, y_{2n+2}, y_{2n+2}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+2}, t), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+3}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t), \mathcal{M}(y_{2n+2}, y_{2n+3}, y_{2n+3}, t)\} \\
&= \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t)
\end{aligned}$$

Therefore $\mathcal{M}(y_{2n+2}, y_{2n+3}, y_{2n+3}, kt) \geq \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t)$.

Hence $\mathcal{M}(y_{n+1}, y_{n+2}, y_{n+2}, kt) \geq \mathcal{M}(y_n, y_{n+1}, y_{n+1}, t)$, for all n.

By lemma 1.14, $\{y_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X.

Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{y_n\}$ converges to the point $z \in X$.

Also, since $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ are subsequences of $\{y_n\}$, they also converge to the point z.

Case I: Since S is continuous, we have $SAx_{2n} \rightarrow Sz$, $SSx_{2n} \rightarrow Sz$.

Also (A, S) is semi compatible, we have $ASx_{2n} \rightarrow Sz$.

Let $x = Sx_{2n}$, $y = x_{2n+1}$ in condition (3) we get

$$\mathcal{M}(ASx_{2n}, Bx_{2n+1}, Bx_{2n+1}, kt) \geq \min \{ \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(SSx_{2n}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(ASx_{2n}, SSx_{2n}, SSx_{2n}, t), \mathcal{M}(ASx_{2n}, Bx_{2n+1}, Bx_{2n+1}, t)\}$$

Taking limit as $n \to \infty$ we get

$$\mathcal{M}(Sz, z, z, kt) \geq \min \{\mathcal{M}(z, z, z, t), \mathcal{M}(Sz, z, z, t), \mathcal{M}(Sz, Sz, Sz, t), \mathcal{M}(Sz, z, z, t) \} = \mathcal{M}(Sz, z, z, t)$$

Therefore by lemma 1.15, $Sz = z$.

Now let $x = z$, $y = x_{2n+1}$ in condition (3) we get

$$\mathcal{M}(Az, Bx_{2n+1}, Bx_{2n+1}, kt) \geq \min \{ \mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Sz, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Az, Sz, Sz, t), \mathcal{M}(Az, Bx_{2n+1}, Bx_{2n+1}, t) \}$$

Taking limit as $n \to \infty$ we get

$$\begin{aligned}\mathcal{M}(Az, z, z, kt) &\geq \min \{ \mathcal{M}(z, z, z, t), \mathcal{M}(Sz, z, z, t), \mathcal{M}(Az, Sz, Sz, t), \mathcal{M}(Az, z, z, t) \} \\ &= \min \{ \mathcal{M}(z, z, z, t), \mathcal{M}(z, z, z, t), \mathcal{M}(Az, z, z, t), \mathcal{M}(Az, z, z, t) \} \\ &= \mathcal{M}(Az, z, z, t)\end{aligned}$$

Therefore by lemma 1.15, $Az = z$.

Therefore $Az = z = Sz$.

Case II: Since T is continuous, we have $TBx_{2n+1} \to Tz$, $TTx_{2n+1} \to Tz$.

Also (B, T) is semi compatible; we have $BTx_{2n+1} \to Tz$.

Let $x = x_{2n}$, $y = Tx_{2n+1}$ in condition (3) we get

$$\mathcal{M}(Ax_{2n}, BTx_{2n+1}, BTx_{2n+1}, kt) \geq \min \{\mathcal{M}(BTx_{2n+1}, TTx_{2n+1}, TTx_{2n+1}, t), \mathcal{M}(Sx_{2n}, TTx_{2n+1}, TTx_{2n+1}, t), \mathcal{M}(Ax_{2n}, Sx_{2n}, Sx_{2n}, t), \mathcal{M}(Ax_{2n}, BTx_{2n+1}, BTx_{2n+1}, t)\}$$

Taking limit as $n \to \infty$ we get

$$\mathcal{M}(z, Tz, Tz, kt) \geq \min \{\mathcal{M}(Tz, Tz, Tz, t), \mathcal{M}(z, Tz, Tz, t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, Tz, Tz, t) \} = \mathcal{M}(z, Tz, Tz, t)$$

Therefore by lemma 1.15, $Tz = z$.

Now let $x = x_{2n}$, $y = z$ in condition (3) we get

$$\mathcal{M}(Ax_{2n}, Bz, Bz, kt) \geq \min \{\mathcal{M}(Bz, Tz, Tz, t), \mathcal{M}(Sx_{2n}, Tz, Tz, t), \mathcal{M}(Ax_{2n}, Sx_{2n}, Sx_{2n}, t), \mathcal{M}(Ax_{2n}, Bz, Bz, t)\}$$

Taking limit as $n \to \infty$ we get

$$\begin{aligned}\mathcal{M}(z, Bz, Bz, kt) &\geq \min \{\mathcal{M}(Bz, Tz, Tz, t), \mathcal{M}(z, Tz, Tz, t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, Bz, Bz, t)\} \\ &= \min \{\mathcal{M}(Bz, z, z, t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, Bz, Bz, t)\} \\ &= \min \{\mathcal{M}(Bz, Bz, z, t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, Bz, Bz, t)\} \\ &= \min \{\mathcal{M}(z, Bz, Bz, t), \mathcal{M}(z, z, z, t) \} \\ &= \mathcal{M}(z, Bz, Bz, t)\end{aligned}$$

Therefore by lemma 1.15, $Bz = z$.

Therefore $Bz = z = Tz$.

Thus we have $Az = Sz = Bz = Tz = z$.

Hence z is a common fixed point of A, B, S, and T.

Uniqueness: Suppose z' ($\neq z$) is another common fixed point of A, B, S, and T.

Now $\mathcal{M}(z, z', z', kt) = \mathcal{M}(Az, Bz', Bz', kt)$
$\geq \min\{\mathcal{M}(Bz', Tz', Tz', t), \mathcal{M}(Sz, Tz', Tz', t), \mathcal{M}(Az, Sz, Sz, t), \mathcal{M}(Az, Bz', Bz', t)\}$
$= \min\{\mathcal{M}(z', z', z', t), \mathcal{M}(z, z', z', t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, z', z', t)\}$
$= \mathcal{M}(z, z', z', t)$

Therefore by lemma 1.15, $z = z'$.

This completes the proof.

Remark: 2.2 Putting $B = A$ in theorem 2.1, we get the following result.

Corollary: 2.3 Let S and T be two continuous self mappings of a complete $\mathcal{M}$ – fuzzy metric space (X, $\mathcal{M}$, *). Let A be a self mapping of X satisfying

(1) $A(X) \subset T(X)$, $A(X) \subset S(X)$.
(2) (A, S) and (A, T) are semi compatible.
(3) there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$\mathcal{M}(Ax, Ay, Ay, kt) \geq \min\{\mathcal{M}(Ay, Ty, Ty, t), \mathcal{M}(Sx, Ty, Ty, t), \mathcal{M}(Ax, Sx, Sx, t), \mathcal{M}(Ax, Ay, Ay, t)\}.$$

Then A, S and T have a unique common fixed point.

Remark: 2.4 Putting $B = A$, $T = S$ in theorem 2.1, we get the following result.

Corollary: 2.5 Let S be continuous self mapping of a complete $\mathcal{M}$ – fuzzy metric space (X, $\mathcal{M}$, *). Let A be a self mapping of X satisfying

(1) $A(X) \subset S(X)$
(2) (A, S) semi compatible pair of mappings
(3) there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$\mathcal{M}(Ax, Ay, Ay, kt) \geq \min\{\mathcal{M}(Ay, Sy, Sy, t), \mathcal{M}(Sx, Sy, Sy, t), \mathcal{M}(Ax, Sx, Sx, t), \mathcal{M}(Ax, Ay, Ay, t)\}.$$

Then A and S have a unique common fixed point.

Remark: 2.6 Putting $B = A$, $T = S = I$ in theorem 2.1, we get the following result.

Corollary: 2.7 Let A be a self mapping of a complete $\mathcal{M}$ – fuzzy metric space (X, $\mathcal{M}$, *) satisfying

$\mathcal{M}(Ax, Ay, Ay, kt) \geq \min\{\mathcal{M}(Ay, y, y, t), \mathcal{M}(x, y, y, t), \mathcal{M}(Ax, x, x, t), \mathcal{M}(Ax, Ay, Ay, t)\}$ for all $x, y \in X$, $t > 0$ and $0 < k < 1$.Then A has a unique fixed point.

Theorem: 2.8 Let A, B, S and T be self mappings of a complete $\mathcal{M}$ – fuzzy metric space (X, $\mathcal{M}$, *) satisfying the following conditions

(1) $A(X) \subset T(X)$, $B(X) \subset S(X)$.

(2) (A, S) and (B, T) are weakly compatible.

(3) there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$\mathcal{M}(Ax, By, By, kt) \geq \min\{\mathcal{M}(By, Ty, Ty, t), \mathcal{M}(Sx, Ty, Ty, t), \mathcal{M}(Ax, Sx, Sx, t), \mathcal{M}(Ax, By, By, t)\}.$$

Then A, B, S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.

Since $A(X) \subset T(X)$, then there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$.

Also, since $B(X) \subset S(X)$, then there exists another point $x_2 \in X$ such that $Bx_1 = Sx_2$.

Then by induction, we can define a sequence $\{y_n\}$ in X such that

$y_{2n+1} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, \ldots$

Now using condition (3) we get

$$\begin{aligned}
\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt) &= \mathcal{M}(Ax_{2n}, Bx_{2n+1}, Bx_{2n+1}, kt) \\
&\geq \min\{\mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Ax_{2n}, Sx_{2n}, Sx_{2n}, t), \\
&\qquad \mathcal{M}(Ax_{2n}, Bx_{2n+1}, Bx_{2n+1}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+2}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n}, y_{2n}, t), \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+2}, t), \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n}, y_{2n}, y_{2n+1}, t), \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t), \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t)\} \\
&= \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t)
\end{aligned}$$

Therefore $\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, kt) \geq \mathcal{M}(y_{2n}, y_{2n+1}, y_{2n+1}, t)$.

Also, $\mathcal{M}(y_{2n+2}, y_{2n+3}, y_{2n+3}, kt) = \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+3}, kt)$

$$\begin{aligned}
&= \mathcal{M}(y_{2n+3}, y_{2n+2}, y_{2n+2}, kt) = \mathcal{M}(Ax_{2n+2}, Bx_{2n+1}, Bx_{2n+1}, kt) \\
&\geq \min\{\mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Sx_{2n+2}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Ax_{2n+2}, Sx_{2n+2}, Sx_{2n+2}, t), \\
&\qquad \mathcal{M}(Ax_{2n+2}, Bx_{2n+1}, Bx_{2n+1}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+2}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+2}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+3}, y_{2n+2}, y_{2n+2}, t), \mathcal{M}(y_{2n+3}, y_{2n+2}, y_{2n+2}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+2}, y_{2n+1}, y_{2n+1}, t), \mathcal{M}(y_{2n+3}, y_{2n+2}, y_{2n+2}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+1}, y_{2n+1}, y_{2n+2}, t), \mathcal{M}(y_{2n+2}, y_{2n+2}, y_{2n+3}, t)\} \\
&= \min\{\mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t), \mathcal{M}(y_{2n+2}, y_{2n+3}, y_{2n+3}, t)\} \\
&= \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t)
\end{aligned}$$

Therefore $\mathcal{M}(y_{2n+2}, y_{2n+3}, y_{2n+3}, kt) \geq \mathcal{M}(y_{2n+1}, y_{2n+2}, y_{2n+2}, t)$.

Hence $\mathcal{M}(y_{n+1}, y_{n+2}, y_{n+2}, kt) \geq \mathcal{M}(y_n, y_{n+1}, y_{n+1}, t)$, for all n.

By lemma 1.14, $\{y_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X.

Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{y_n\}$ converges to the point $z \in X$.

Also, since $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ are subsequences of $\{y_n\}$, they also converge to the point z.

Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $z = Su$.

Then by condition (3) we have

$$\begin{aligned}
\mathcal{M}(Au, Bx_{2n+1}, Bx_{2n+1}, kt) \geq \min\{&\mathcal{M}(Bx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(Su, Tx_{2n+1}, Tx_{2n+1}, t), \\
&\mathcal{M}(Au, Su, Su, t), \mathcal{M}(Au, Bx_{2n+1}, Bx_{2n+1}, t)\}
\end{aligned}$$

Taking limit as $n \to \infty$ we get

$$\begin{aligned}
\mathcal{M}(Au, z, z, kt) &\geq \min\{\mathcal{M}(z, z, z, t), \mathcal{M}(Su, z, z, t), \mathcal{M}(Au, Su, Su, t), \mathcal{M}(Au, z, z, t)\} \\
&= \min\{\mathcal{M}(z, z, z, t), \mathcal{M}(z, z, z, t), \mathcal{M}(Au, z, z, t), \mathcal{M}(Au, z, z, t)\} \\
&= \mathcal{M}(Au, z, z, t).
\end{aligned}$$

Therefore by lemma l.15, $Au = z$.

Therefore $Au = z = Su$.

Similarly, since $A(X) \subset T(X)$, there exists a point $v \in X$ such that $z = Tv$.

Then by condition (3) we have

$$\begin{aligned}
\mathcal{M}(z, Bv, Bv, kt) &= \mathcal{M}(Au, Bv, Bv, kt) \\
&\geq \min\{\mathcal{M}(Bv, Tv, Tv, t), \mathcal{M}(Su, Tv, Tv, t), \mathcal{M}(Au, Su, Su, t), \mathcal{M}(Au, Bv, Bv, t)\} \\
&= \min\{\mathcal{M}(Bv, z, z, t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, Bv, Bv, t)\} \\
&= \mathcal{M}(z, Bv, Bv, t).
\end{aligned}$$

Therefore by lemma l.15, $Bv = z$.

Therefore $Bv = z = Tv$.

Hence $Au = z = Su = Bv = Tv$.

Since the pair of mappings (A, S) is weakly compatible, so $ASu = SAu$ gives $Az = Sz$.

Now we prove z is a fixed point of A.

$$\begin{aligned}\mathcal{M}(Az, z, z, kt) &= \mathcal{M}(Az, Bv, Bv, kt)\\ &\geq \min\{\mathcal{M}(Bv, Tv, Tv, t), \mathcal{M}(Sz, Tv, Tv, t), \mathcal{M}(Az, Sz, Sz, t), \mathcal{M}(Az, Bv, Bv, t)\}\\ &= \min\{\mathcal{M}(z, z, z, t), \mathcal{M}(Az, z, z, t), \mathcal{M}(Az, Az, Az, t), \mathcal{M}(Az, z, z, t)\}\\ &= \mathcal{M}(Az, z, z, t).\end{aligned}$$

Therefore by lemma 1.15, $Az = z$.

Hence $Az = z = Sz$.

Since the pair of mappings (B, T) is weakly compatible, so $BTv = TBv$ gives $Bz = Tz$.

Now we prove z is a fixed point of B.

$$\begin{aligned}\mathcal{M}(z, Bz, Bz, kt) &= \mathcal{M}(Az, Bz, Bz, kt)\\ &\geq \min\{\mathcal{M}(Bz, Tz, Tz, t), \mathcal{M}(Sz, Tz, Tz, t), \mathcal{M}(Az, Sz, Sz, t), \mathcal{M}(Az, Bz, Bz, t)\}\\ &= \min\{\mathcal{M}(Bz, Bz, Bz, t), \mathcal{M}(z, Bz, Bz, t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, Bz, Bz, t)\}\\ &= \mathcal{M}(z, Bz, Bz, t).\end{aligned}$$

Therefore by lemma 1.15, $Bz = z$.

Hence $Bz = z = Tz$.

Thus we have $Az = Bz = Sz = Tz = z$.

Hence z is a common fixed point of A, B, S and T.

Uniqueness: Suppose z' ($\neq z$) is another common fixed point of A, B, S, and T.

Now
$$\begin{aligned}\mathcal{M}(z, z', z', kt) &= \mathcal{M}(Az, Bz', Bz', kt)\\ &\geq \min\{\mathcal{M}(Bz', Tz', Tz', t), \mathcal{M}(Sz, Tz', Tz', t), \mathcal{M}(Az, Sz, Sz, t), \mathcal{M}(Az, Bz', Bz', t)\}\\ &= \min\{\mathcal{M}(z', z', z', t), \mathcal{M}(z, z', z', t), \mathcal{M}(z, z, z, t), \mathcal{M}(z, z', z', t)\}\\ &= \mathcal{M}(z, z', z', t)\end{aligned}$$

Therefore by lemma 1.15, $z = z'$.

This completes the proof.

Remark: 2.9 Putting $B = A$ in theorem 2.8, we get the following result.

Corollary: 2.10 Let A, S and T be self mappings of a complete $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$ satisfying the following conditions

(1) $A(X) \subset T(X)$, $A(X) \subset S(X)$.
(2) (A, S) and (A, T) are weakly compatible.
(3) there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,
$$\mathcal{M}(Ax, Ay, Ay, kt) \geq \min\{\mathcal{M}(Ay, Ty, Ty, t), \mathcal{M}(Sx, Ty, Ty, t), \mathcal{M}(Ax, Sx, Sx, t), \mathcal{M}(Ax, Ay, Ay, t)\}.$$

Then A, S and T have a unique common fixed point.

Remark: 2.11 Putting $B = A$, $T = S$ in theorem 2.8, we get the following result.

Corollary: 2.12 Let A and S be self mappings of a complete $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$ satisfying the following conditions

(1) $A(X) \subset S(X)$.
(2) (A, S) weakly compatible pair of mappings.
(3) there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$\mathcal{M}(Ax, Ay, Ay, kt) \geq \min\{\mathcal{M}(Ay, Sy, Sy, t), \mathcal{M}(Sx, Sy, Sy, t), \mathcal{M}(Ax, Sx, Sx, t), \mathcal{M}(Ax, Ay, Ay, t)\}.$$

Then A and S have a unique common fixed point.

Remark: 2.13 Putting $B = A$, $T = S = I$ in theorem 2.8, we get the following result.

Corollary: 2.14 Let A be a self mapping of a complete $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$ satisfying

$$\mathcal{M}(Ax, Ay, Ay, kt) \geq \min\{\mathcal{M}(Ay, y, y, t), \mathcal{M}(x, y, y, t), \mathcal{M}(Ax, x, x, t), \mathcal{M}(Ax, Ay, Ay, t)\}$$

for all $x, y \in X$, $t > 0$ and $0 < k < 1$.Then A has a unique fixed point.

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