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SOME COMMON FIXED POINT THEOREMS IN $\mathcal{M}$ - FUZZY METRIC SPACE

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ABSTRACT

In this paper we prove some common fixed point theorems for generalized contraction mappings in complete $\mathcal{M}$ – fuzzy metric space. Also we prove some common fixed point theorems for two weakly compatible mappings in $\mathcal{M}$ – fuzzy metric space.

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Keywords: *Complete $\mathcal{M}$ – Fuzzy metric space, $\mathcal{M}$ – Fuzzy contraction mapping, weakly compatible mapping, Common fixed point.*

1. INTRODUCTION AND PRELIMINARIES

The concept of fuzzy sets was introduced by Zadeh [14] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. Fixed point theorems in fuzzy mathematics are emerging with vigorous hope and vital trust. It appears that Kramosil and Michalek's study of fuzzy metric spaces paves a way for very soothing machinery to develop fixed point theorems for contractive type maps. Kramosil and Michalek [6] introduced the concept of fuzzy metric space and modified by George and Veeramani [2]. Many authors [4, 7] have proved fixed point theorems in fuzzy metric space. Recently Sedghi and Shobe [12] introduced D^* - metric space as a probable modification of the definition of D - metric introduced by Dhage [1], and prove some basic properties in D^* - metric spaces. Using D^* - metric concepts, Sedghi and Shobe define $\mathcal{M}$ – fuzzy metric space and proved a common fixed point theorem in it. In this paper we prove some common fixed point theorems for generalized contraction mappings in complete $\mathcal{M}$ – fuzzy metric space. Also we prove some common fixed point theorems for two weakly compatible mappings in $\mathcal{M}$ – fuzzy metric space.

Definition: 1.1 Let X be a nonempty set. A generalized metric (or D^* - metric) on X is a function: $D^* : X^3 \to [0, \infty)$, that satisfies the following conditions for each $x, y, z, a \in X$

(i) $D^*(x, y, z) \geq 0$,
(ii) $D^*(x, y, z) = 0$ iff $x = y = z$,
(iii) $D^*(x, y, z) = D^*(p\{x, y, z\})$, (symmetry) where p is a permutation function,
(iv) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D^*), is called a generalized metric (or D^* - metric) space.

Example: 1.2 Examples of D* - metric are
(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,
(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.
Here, d is the ordinary metric on X.

Definition: 1.3 A fuzzy set $\mathcal{M}$ in an arbitrary set X is a function with domain X and values in [0, 1].

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Definition: 1.4 A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions

(i) $*$ is associative and commutative,

(ii) $*$ is continuous,

(iii) $a * 1 = a$ for all $a \in [0, 1]$,

(iv) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Examples for continuous t-norm are $a*b = ab$ and $a*b = \min\{a, b\}$.

Definition: 1.5 A 3-tuple $(X, \mathcal{M}, *)$ is called $\mathcal{M}$ – fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$

$(FM-1)$ $\mathcal{M}(x, y, z, t) > 0$

$(FM-2)$ $\mathcal{M}(x, y, z, t) = 1$ iff $x = y = z$

$(FM-3)$ $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function

$(FM-4)$ $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t+s)$

$(FM-5)$ $\mathcal{M}(x, y, z, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous

$(FM-6)$ $\lim_{t \to \infty} \mathcal{M}(x, y, z, t) = 1$.

Example: 1.6 Let X be a nonempty set and D^* is the D^* - metric on X. Denote $a*b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}$$

for all $x, y, z \in X$, then $(X, \mathcal{M}, *)$ is a $\mathcal{M}$- fuzzy metric space. We call this $\mathcal{M}$-fuzzy metric induced by D^* - metric space. Thus every D^* - metric induces a $\mathcal{M}$-fuzzy metric.

Lemma: 1.7 ([12]) Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. Then for every $t > 0$ and for every $x, y \in X$. we have $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

Lemma: 1.8 ([12]) Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ –fuzzy metric space. Then $\mathcal{M}(x, y, z, t)$ is non-decreasing with respect to t, for all x, y, z in X.

Definition: 1.9 Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. For $t > 0$, the open ball $B_{\mathcal{M}}(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$B_{\mathcal{M}}(x, r, t) = \{y \in X: \mathcal{M}(x, y, y, t) > 1 - r\}$.

A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B_{\mathcal{M}}(x, r, t) \subseteq A$.

Definition: 1.10 Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ – fuzzy metric space and $\{x_n\}$ be a sequence in X

(a) $\{x_n\}$ is said to be converges to a point $x \in X$ if $\lim_{n \to \infty} \mathcal{M}(x, x, x_n, t) = 1$ for all $t > 0$

(b) $\{x_n\}$ is called Cauchy sequence if $\lim_{n \to \infty} \mathcal{M}(x_{n+p}, x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$

(c) A $\mathcal{M}$ – fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Remark: 1.11 Since $*$ is continuous, it follows from (FM-4) that the limit of the sequence is uniquely determined.

Definition: 1.12 Let S and T be two self mappings of a $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$. Then the mappings S and T are said to be weakly compatible if they commute at their coincidence points; that is, if $Sx = Tx$ for some $x \in X$, then $STx = TSx$.

Definition: 1.13 Let A and B be two self mappings of a $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$. We say that A and B satisfy the property (E), if there exists a sequence $\{x_n\}$ such that $\lim_{n \to \infty} \mathcal{M}(Ax_n, u, u, t) = \lim_{n \to \infty} \mathcal{M}(Bx_n, u, u, t) = 1$, for some $u \in X$ and $t > 0$.

Example: 1.14 Let $X = \mathbb{R}$ and $\mathcal{M}(x, y, z, t) = \frac{t}{t+|x-y|+|y-z|+|x-z|}$ for every $x, y, z \in X$ and $t > 0$. Let A and B defined by $Ax = 2x$, $Bx = x + 1$.

Consider the sequence $x_n = \frac{1}{n} + 1$, $n = 1, 2, 3,...$ Then we have

$\lim_{n \to \infty} \mathcal{M}(Ax_n, 2, 2, t) = \lim_{n \to \infty} \mathcal{M}(Bx_n, 2, 2, t) = 1$ for every $t > 0$. Then A and B satisfy the property (E).

Lemma: 1.15 ([10]) Let $\{x_n\}$ be a sequence in a $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$ with the condition (*FM-6*). If there exists a number $k \in (0, 1)$ such that

$$\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t) \geq \mathcal{M}(x_{n-1}, x_n, x_n, \frac{t}{k})$$

for all $t > 0$ and $n = 1, 2, 3 \ldots$, then $\{x_n\}$ is a Cauchy sequence.

Lemma 1.16 ([10]) Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ – fuzzy metric space with condition (*FM-6*). If there exists a number $k \in (0, 1)$ such that $\mathcal{M}(x, y, z, kt) \geq \mathcal{M}(x, y, z, t)$, for all $x, y, z \in X$ and $t > 0$, then $x = y = z$.

MAIN RESULTS:

Theorem: 2.1 Let S and T be two self mappings of a complete $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$. If there exists a number $k \in (0, 1)$ such that

$$\mathcal{M}(Sx, Ty, Ty, kt) \geq \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, x, Sx, t), \mathcal{M}(y, y, Ty, t)\}$$

for all $x, y \in X$ and $t > 0$, then S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.
Define a sequence $\{x_n\}$ in X as $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for $n = 0, 1, 2, \ldots$

Now we prove that $\{x_n\}$ is a Cauchy sequence in X.

For $n \geq 0$, we have

$$\begin{aligned}
\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt) &= \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, kt)\\
&\geq \min\{\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x_{2n}, x_{2n}, Sx_{2n}, t), \mathcal{M}(x_{2n+1}, x_{2n+1}, Tx_{2n+1}, t)\}\\
&= \min\{\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x_{2n}, x_{2n}, x_{2n+1}, t), \mathcal{M}(x_{2n+1}, x_{2n+1}, x_{2n+2}, t)\}\\
&= \min\{\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)\}\\
&= \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)
\end{aligned}$$

Therefore, $\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt) \geq \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)$.

Also,

$$\begin{aligned}
\mathcal{M}(x_{2n+2}, x_{2n+3}, x_{2n+3}, kt) &= \mathcal{M}(x_{2n+2}, x_{2n+2}, x_{2n+3}, kt)\\
&= \mathcal{M}(Tx_{2n+1}, Tx_{2n+1}, Sx_{2n+2}, kt) = \mathcal{M}(Sx_{2n+2}, Tx_{2n+1}, Tx_{2n+1}, kt)\\
&\geq \min\{\mathcal{M}(x_{2n+2}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x_{2n+2}, x_{2n+2}, Sx_{2n+2}, t), \mathcal{M}(x_{2n+1}, x_{2n+1}, Tx_{2n+1}, t)\}\\
&= \min\{\mathcal{M}(x_{2n+1}, x_{2n+1}, x_{2n+2}, t), \mathcal{M}(x_{2n+2}, x_{2n+2}, x_{2n+3}, t), \mathcal{M}(x_{2n+1}, x_{2n+1}, x_{2n+2}, t)\}\\
&= \min\{\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t), \mathcal{M}(x_{2n+2}, x_{2n+3}, x_{2n+3}, t), \mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)\}\\
&= \mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)
\end{aligned}$$

Therefore, $\mathcal{M}(x_{2n+2}, x_{2n+3}, x_{2n+3}, kt) \geq \mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)$.

Hence $\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, kt) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)$, for all n.

By lemma 1.15, $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X.

Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{x_n\}$ converges to a point $x \in X$.

Now we prove that x is a common fixed point of S and T.

Now consider

$$\begin{aligned}
\mathcal{M}(Sx, x, x, kt) &= \lim_{n\to\infty} \mathcal{M}(Sx, x_{2n+2}, x_{2n+2}, kt)\\
&= \lim_{n\to\infty} \mathcal{M}(Sx, Tx_{2n+1}, Tx_{2n+1}, kt)\\
&\geq \lim_{n\to\infty} \min\{\mathcal{M}(x, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x, x, Sx, t), \mathcal{M}(x_{2n+1}, x_{2n+1}, Tx_{2n+1}, t)\}\\
&= \lim_{n\to\infty} \min\{\mathcal{M}(x, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x, x, Sx, t), \mathcal{M}(x_{2n+1}, x_{2n+1}, x_{2n+2}, t)\}\\
&= \min\{\mathcal{M}(x, x, x, t), \mathcal{M}(x, x, Sx, t), \mathcal{M}(x, x, x, t)\}\\
&= \mathcal{M}(x, x, Sx, t)\\
&= \mathcal{M}(Sx, x, x, t)
\end{aligned}$$

Hence $\mathcal{M}(Sx, x, x, kt) \geq \mathcal{M}(Sx, x, x, t)$, for each $t > 0$.

Therefore by lemma 1.16, $Sx = x$.

Similarly, $Tx = x$. Hence x is a common fixed point of S and T.

Uniqueness: Let y ($\neq x$) be another common fixed point of S and T.

Now consider

$\mathcal{M}(x, y, y, kt) = \mathcal{M}(Sx, Ty, Ty, kt)$

$\geq \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, x, Sx, t), \mathcal{M}(y, y, Ty, t)\}$

$= \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, x, x, t), \mathcal{M}(y, y, y, t)\}$

$= \mathcal{M}(x, y, y, t)$.

Hence $\mathcal{M}(x, y, y, kt) \geq \mathcal{M}(x, y, y, t)$, for each $t > 0$.

Therefore by lemma 1.16, $x = y$.

This completes the proof.

Example: 2.2 Let $X = [0, 1]$ and $\mathcal{M}(x, y, z, t) = \frac{t}{t+|x-y|+|y-z|+|x-z|}$ for every $x, y, z \in X$ and $t > 0$. S and T be two self mappings of X defined by $Sx = \dfrac{2x}{5}$, $Tx = \dfrac{x}{5}$. Then S and T satisfy the condition of the above theorem and have a common fixed point at $x = 0$.

Remark: 2.3 Putting $T = S$ in the above theorem, we get the following corollary 2.4

Corollary: 2.4 Let $(X, \mathcal{M}, *)$ complete $\mathcal{M}$ – fuzzy metric space and let $S: X \to X$ be a mapping such that
$\mathcal{M}(Sx, Sy, Sy, kt) \geq \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, x, Sx, t), \mathcal{M}(y, y, Sy, t)\}$ for all $x, y \in X$, $t > 0$ and $0 < k < 1$.Then S has a unique fixed point.

Theorem: 2.5 Let S and T be two self mappings of a complete $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$ with continuous t-norm * defined by $a*b = \min\{a, b\}$ for all $a, b \in [0, 1]$. If there exists a number $k \in (0, 1)$ such that

$\mathcal{M}(Sx, Ty, Ty, kt) \geq \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, x, Sx, t), \mathcal{M}(y, y, Ty, t), \mathcal{M}(x, x, Ty, 2t)\}$

for all $x, y \in X$ and $t > 0$, then S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.
Define a sequence $\{x_n\}$ in X as $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for $n = 0, 1, 2, \ldots$

Now we prove that $\{x_n\}$ is a Cauchy sequence in X.

For $n \geq 0$, we have

$\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt) = \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, kt)$

$\geq \min\{\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x_{2n}, x_{2n}, Sx_{2n}, t), \mathcal{M}(x_{2n+1}, x_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(x_{2n}, x_{2n}, Tx_{2n+1}, 2t)\}$

$= \min\{\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x_{2n}, x_{2n}, x_{2n+1}, t), \mathcal{M}(x_{2n+1}, x_{2n+1}, x_{2n+2}, t), \mathcal{M}(x_{2n}, x_{2n}, x_{2n+2}, 2t)\}$

$= \min\{\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t), \mathcal{M}(x_{2n}, x_{2n}, x_{2n+2}, 2t)\}$

$\geq \min\{\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t), \mathcal{M}(x_{2n}, x_{2n}, x_{2n+1}, t)*\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)\}$

$= \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)*\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)$

Therefore, $\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt) \geq \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)*\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)$
which implies that

$\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt) \geq \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)$

Similarly, we prove that $\mathcal{M}(x_{2n+2}, x_{2n+3}, x_{2n+3}, kt) \geq \mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)$

Hence $\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, kt) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)$, for all n.

Therefore, $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X.

Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{x_n\}$ converges to a point $x \in X$.

Now we prove that x is a common fixed point of S and T.

Now consider

$\mathcal{M}(Sx, x, x, kt) = lim_{n \to \infty}\ \mathcal{M}(Sx, x_{2n+2}, x_{2n+2}, kt)$

$= lim_{n \to \infty}\ \mathcal{M}(Sx, Tx_{2n+1}, Tx_{2n+1}, kt)$

$\geq lim_{n\to\infty}$ min { $\mathcal{M}(x, x_{2n+1}, x_{2n+1}, t)$, $\mathcal{M}(x, x, Sx, t)$, $\mathcal{M}(x_{2n+1}, x_{2n+1}, Tx_{2n+1}, t)$, $\mathcal{M}(x, x, Tx_{2n+1}, 2t)$}
$= lim_{n\to\infty}$ min { $\mathcal{M}(x, x_{2n+1}, x_{2n+1}, t)$, $\mathcal{M}(x, x, Sx, t)$, $\mathcal{M}(x_{2n+1}, x_{2n+1}, x_{2n+2}, t)$, $\mathcal{M}(x, x, x_{2n+2}, 2t)$}
= min { $\mathcal{M}(x, x, x, t)$, $\mathcal{M}(x, x, Sx, t)$, $\mathcal{M}(x, x, x, t)$, $\mathcal{M}(x, x, x, 2t)$}
$= \mathcal{M}(x, x, Sx, t)$
$= \mathcal{M}(Sx, x, x, t)$

Hence $\mathcal{M}(Sx, x, x, kt) \geq \mathcal{M}(Sx, x, x, t)$, for each $t > 0$.

Therefore, $Sx = x$.

Similarly, $Tx = x$. Hence x is a common fixed point of S and T.

Uniqueness: Let y ($\neq x$) be another common fixed point of S and T.
Now consider
$\mathcal{M}(x, y, y, kt) = \mathcal{M}(Sx, Ty, Ty, kt)$
$\geq$ min {$\mathcal{M}(x, y, y, t)$, $\mathcal{M}(x, x, Sx, t)$, $\mathcal{M}(y, y, Ty, t)$, $\mathcal{M}(x, x, Ty, 2t)$}
= min {$\mathcal{M}(x, y, y, t)$, $\mathcal{M}(x, x, x, t)$, $\mathcal{M}(y, y, y, t)$, $\mathcal{M}(x, x, y, 2t)$}
= min {$\mathcal{M}(x, y, y, t)$, 1, 1, $\mathcal{M}(x, y, y, 2t)$}
$= \mathcal{M}(x, y, y, t)$

Hence $\mathcal{M}(x, y, y, kt) \geq \mathcal{M}(x, y, y, t)$, for each $t > 0$.

Therefore, $x = y$.

This completes the proof.

Example: 2.6 Let $X = [0, 1]$ and $a*b = \min\{a, b\}$. Let $\mathcal{M}(x, y, z, t) = \frac{t}{t+|x-y|+|y-z|+|x-z|}$ for every $x, y, z \in X$ and $t > 0$. S and T be two self mappings of X defined by $Sx = \frac{x}{2}$, $Tx = \frac{x}{4}$. Then S and T satisfy the condition of the above theorem and have a common fixed point at $x = 0$.

Remark: 2.7 Putting $T = S$ in the above theorem, we get the following corollary 2.8

Corollary: 2.8 Let $(X, \mathcal{M}, *)$ complete $\mathcal{M}$ – fuzzy metric space with continuous t-norm * defined by $a*b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and let $S: X \to X$ be a mapping such that
$\mathcal{M}(Sx, Sy, Sy, kt) \geq \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, x, Sx, t), \mathcal{M}(y, y, Sy, t), \mathcal{M}(x, x, Sy, 2t)\}$
for all $x, y \in X$, $t > 0$ and $0 < k < 1$.Then S has a unique fixed point.

Theorem: 2.9 Let S and T be two self mappings of a complete $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$ with continuous t-norm * defined by $a*b = \min\{a, b\}$ for all $a, b \in [0, 1]$. If there exists a number $k \in (0, 1)$ such that
$\mathcal{M}(Sx, Ty, Ty, kt) \geq \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, x, Sx, t)*\mathcal{M}(y, y, Ty, t), \mathcal{M}(x, x, Ty, 2t)\}$
for all $x, y \in X$ and $t > 0$, then S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.
Define a sequence $\{x_n\}$ in X as $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for $n = 0, 1, 2, \ldots$

Now we prove that $\{x_n\}$ is a Cauchy sequence in X.

For $n \geq 0$, we have
$\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt) = \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, kt)$
$\geq$ min { $\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)$, $\mathcal{M}(x_{2n}, x_{2n}, Sx_{2n}, t)*\mathcal{M}(x_{2n+1}, x_{2n+1}, Tx_{2n+1}, t)$, $\mathcal{M}(x_{2n}, x_{2n}, Tx_{2n+1}, 2t)$}
= min { $\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)$, $\mathcal{M}(x_{2n}, x_{2n}, x_{2n+1}, t)*\mathcal{M}(x_{2n+1}, x_{2n+1}, x_{2n+2}, t)$, $\mathcal{M}(x_{2n}, x_{2n}, x_{2n+2}, 2t)$}
= min { $\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)$, $\mathcal{M}(x_{2n}, x_{2n}, x_{2n+1}, t)*\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)$, $\mathcal{M}(x_{2n}, x_{2n}, x_{2n+2}, 2t)$}
$\geq$ min { $\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)$, $\mathcal{M}(x_{2n}, x_{2n}, x_{2n+1}, t)*\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)$,
$\mathcal{M}(x_{2n}, x_{2n}, x_{2n+1}, t)*\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)$}
$= \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)*\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)$

Therefore, $\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt) \geq \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)*\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)$ which implies that
$\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt) \geq \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)$

Similarly, we prove that $\mathcal{M}(x_{2n+2}, x_{2n+3}, x_{2n+3}, kt) \geq \mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)$

Hence $\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, kt) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)$, for all n.

Therefore, $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X.

Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{x_n\}$ converges to a point $x \in X$.

Now we prove that x is a common fixed point of S and T.

Now consider

$$\begin{aligned}
\mathcal{M}(Sx, x, x, kt) &= \lim_{n\to\infty} \mathcal{M}(Sx, x_{2n+2}, x_{2n+2}, kt)\\
&= \lim_{n\to\infty} \mathcal{M}(Sx, Tx_{2n+1}, Tx_{2n+1}, kt)\\
&\geq \lim_{n\to\infty} \min\{\mathcal{M}(x, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x, x, Sx, t)*\mathcal{M}(x_{2n+1}, x_{2n+1}, Tx_{2n+1}, t), \mathcal{M}(x, x, Tx_{2n+1}, 2t)\}\\
&= \lim_{n\to\infty} \min\{\mathcal{M}(x, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x, x, Sx, t)*\mathcal{M}(x_{2n+1}, x_{2n+1}, x_{2n+2}, t), \mathcal{M}(x, x, x_{2n+2}, 2t)\}\\
&= \min\{\mathcal{M}(x, x, x, t), \mathcal{M}(x, x, Sx, t)*\mathcal{M}(x, x, x, t), \mathcal{M}(x, x, x, 2t)\}\\
&= \min\{1, \mathcal{M}(x, x, Sx, t)*1, 1\}\\
&= \min\{1, \mathcal{M}(x, x, Sx, t), 1\}\\
&= \mathcal{M}(x, x, Sx, t)\\
&= \mathcal{M}(Sx, x, x, t)
\end{aligned}$$

Hence $\mathcal{M}(Sx, x, x, kt) \geq \mathcal{M}(Sx, x, x, t)$, for each $t > 0$.

Therefore, $Sx = x$.

Similarly, $Tx = x$. Hence x is a common fixed point of S and T.

Uniqueness: Let y ($\neq x$) be another common fixed point of S and T.

Now consider

$$\begin{aligned}
\mathcal{M}(x, y, y, kt) &= \mathcal{M}(Sx, Ty, Ty, kt)\\
&\geq \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, x, Sx, t)*\mathcal{M}(y, y, Ty, t), \mathcal{M}(x, x, Ty, 2t)\}\\
&= \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, x, x, t)*\mathcal{M}(y, y, y, t), \mathcal{M}(x, x, y, 2t)\}\\
&= \min\{\mathcal{M}(x, y, y, t), 1*1, \mathcal{M}(x, y, y, 2t)\}\\
&= \min\{\mathcal{M}(x, y, y, t), 1, \mathcal{M}(x, y, y, 2t)\}\\
&= \mathcal{M}(x, y, y, t)
\end{aligned}$$

Hence $\mathcal{M}(x, y, y, kt) \geq \mathcal{M}(x, y, y, t)$, for each $t > 0$.

Therefore, $x = y$.

This completes the proof.

Example: 2.10 Let $X = [0, 1]$ and $a*b = \min\{a, b\}$. Let $\mathcal{M}(x, y, z, t) = \frac{t}{t+|x-y|+|y-z|+|x-z|}$ for every $x, y, z \in X$ and $t > 0$. S and T be two self mappings of X defined by $Sx = \frac{x}{4}$, $Tx = \frac{x}{8}$. Then S and T satisfy the condition of the above theorem and have a common fixed point at $x = 0$.

Remark: 2.11 Putting $T = S$ in the above theorem, we get the following corollary 2.12

Corollary: 2.12 Let $(X, \mathcal{M}, *)$ complete $\mathcal{M}$ – fuzzy metric space with continuous t-norm * defined by $a*b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and let $S: X \to X$ be a mapping such that

$$\mathcal{M}(Sx, Sy, Sy, kt) \geq \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, x, Sx, t)*\mathcal{M}(y, y, Sy, t), \mathcal{M}(x, x, Sy, 2t)\}$$

for all $x, y \in X$, $t > 0$ and $0 < k < 1$.Then S has a unique fixed point.

Theorem: 2.13 Let S and T be two self mappings of a complete $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$ with continuous t-norm * defined by $a*b = \min\{a, b\}$ for all $a, b \in [0, 1]$. If there exists a number $k \in (0, 1)$ such that

$$\mathcal{M}(Sx, Ty, Ty, kt) \geq \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, Sx, Ty, 2t)\}$$

for all $x, y \in X$ and $t > 0$, then S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.
Define a sequence $\{x_n\}$ in X as $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for $n = 0, 1, 2, \ldots$

Now we prove that $\{x_n\}$ is a Cauchy sequence in X.

For $n \geq 0$, we have

$\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt) = \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, kt)$

$\geq \min\{\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x_{2n}, Sx_{2n}, Tx_{2n+1}, 2t)\}$

$= \min\{\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+2}, 2t)\}$

$\geq \min\{\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)*\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)\}$

$= \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)*\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)$

Therefore, $\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt) \geq \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)*\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)$
which implies that

$\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt) \geq \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)$

Similarly, we prove that $\mathcal{M}(x_{2n+2}, x_{2n+3}, x_{2n+3}, kt) \geq \mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)$

Hence $\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, kt) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t)$, for all n.

Therefore, $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X.

Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{x_n\}$ converges to a point $x \in X$.

Now we prove that x is a common fixed point of S and T.

Now consider

$\mathcal{M}(Sx, x, x, kt) = lim_{n \to \infty}\ \mathcal{M}(Sx, x_{2n+2}, x_{2n+2}, kt)$

$= lim_{n \to \infty}\ \mathcal{M}(Sx, Tx_{2n+1}, Tx_{2n+1}, kt)$

$\geq lim_{n \to \infty} \min\{\mathcal{M}(x, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x, Sx, Tx_{2n+1}, 2t)\}$

$= lim_{n \to \infty} \min\{\mathcal{M}(x, x_{2n+1}, x_{2n+1}, t), \mathcal{M}(x, Sx, x_{2n+2}, 2t)\}$

$= \min\{\mathcal{M}(x, x, x, t), \mathcal{M}(x, Sx, x, 2t)\}$

$= \mathcal{M}(Sx, x, x, 2t)$

$\geq \mathcal{M}(Sx, x, x, t)$

Hence $\mathcal{M}(Sx, x, x, kt) \geq \mathcal{M}(Sx, x, x, t)$, for each $t > 0$.

Therefore, $Sx = x$.

Similarly, $Tx = x$. Hence x is a common fixed point of S and T.

Uniqueness: Let y ($\neq x$) be another common fixed point of S and T.

Now consider

$\mathcal{M}(x, y, y, kt) = \mathcal{M}(Sx, Ty, Ty, kt)$

$\geq \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, Sx, Ty, 2t)\}$

$= \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, x, y, 2t)\}$

$= \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, y, y, 2t)\}$

$= \mathcal{M}(x, y, y, t)$

Hence $\mathcal{M}(x, y, y, kt) \geq \mathcal{M}(x, y, y, t)$, for each $t > 0$.

Therefore, $x = y$.

This completes the proof.

Example: 2.14 Let $X = [0, 1]$ and $a*b = \min\{a, b\}$. Let $\mathcal{M}(x, y, z, t) = \frac{t}{t+|x-y|+|y-z|+|x-z|}$ for every $x, y, z \in X$ and $t > 0$. S and T be two self mappings of X defined by $Sx = \frac{x+1}{2}$, $Tx = \frac{x+3}{4}$. Then S and T satisfy the condition of the above theorem and have a common fixed point at $x = 1$.

Remark: 2.15 Putting $T = S$ in the above theorem, we get the following corollary 2.16

Corollary: 2.16 Let $(X, \mathcal{M}, *)$ complete $\mathcal{M}$ – fuzzy metric space with continuous t-norm $*$ defined by $a*b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and let $S: X \to X$ be a mapping such that

$$\mathcal{M}(Sx, Sy, Sy, kt) \geq \min\{\mathcal{M}(x, y, y, t), \mathcal{M}(x, Sx, Sy, 2t)\}$$

for all $x, y \in X$, $t > 0$ and $0 < k < 1$.Then S has a unique fixed point.

Theorem: 2.17 Let S and T be two weakly compatible self mappings of a $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$ such that

(i) S and T satisfy the property (E).

(ii) For every $x, y, z \in X$ and $t > 0$, with $x \neq y$ or $y \neq z$ or $z \neq x$

$$\mathcal{M}(Tx, Ty, Tz, kt) \geq \min\{\mathcal{M}(Sx, Sy, Sz, t), \mathcal{M}(Tx, Ty, Tz, t), \mathcal{M}(Tx, Sx, Sx, t), \mathcal{M}(Ty, Sy, Sy, t), \mathcal{M}(Tx, Sy, Sz, t), \mathcal{M}(Sx, Ty, Tz, t)\}, \text{ where } 0 < k < 1.$$

(iii) $T(X) \subset S(X)$.

(iv) $T(X)$ or $S(X)$ is complete subspace of X.

Then S and T have a unique common fixed point.

Proof: Since S and T satisfy the property (E), there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} Sx_n = lim_{n\to\infty} Tx_n = x_0$, for some $x_0 \in X$.

Since $x_0 \in T(X)$ and $T(X) \subset S(X)$, there exists some point a in X such that $x_0 = Sa$, where $x_0 = lim_{n\to\infty} Sx_n$

We claim that $Ta = Sa$. Suppose that $Ta \neq Sa$.

Condition *(ii)* implies that

$$\mathcal{M}(Tx_n, Ta, Ta, kt) \geq \min\{\mathcal{M}(Sx_n, Sa, Sa, t), \mathcal{M}(Tx_n, Ta, Ta, t), \mathcal{M}(Tx_n, Sx_n, Sx_n, t), \mathcal{M}(Ta, Sa, Sa, t), \mathcal{M}(Tx_n, Sa, Sa, t), \mathcal{M}(Sx_n, Ta, Ta, t)\}$$

Letting $n \to \infty$ we get

$$\begin{aligned}\mathcal{M}(Sa, Ta, Ta, kt) &\geq \min\{\mathcal{M}(Sa, Sa, Sa, t), \mathcal{M}(Sa, Ta, Ta, t), \mathcal{M}(Sa, Sa, Sa, t), \mathcal{M}(Ta, Sa, Sa, t), \mathcal{M}(Sa, Sa, Sa, t), \mathcal{M}(Sa, Ta, Ta, t)\}\\ &= \min\{1, \mathcal{M}(Sa, Ta, Ta, t), 1, \mathcal{M}(Ta, Ta, Sa, t), 1, \mathcal{M}(Sa, Ta, Ta, t)\}\\ &= \mathcal{M}(Sa, Ta, Ta, t)\end{aligned}$$

Thus, $\mathcal{M}(Sa, Ta, Ta, kt) \geq \mathcal{M}(Sa, Ta, Ta, t)$, for all $t > 0$.

Therefore by lemma 1.16, $Ta = Sa$.

Since S and T are weakly compatible, $STa = TSa = SSa = TTa$.

Now we show that Ta is common fixed point of S and T. Suppose that $Ta \neq TTa$.

$$\begin{aligned}\mathcal{M}(Ta, TTa, TTa, kt) &\geq \min\{\mathcal{M}(Sa, STa, STa, t), \mathcal{M}(Ta, TTa, TTa, t), \mathcal{M}(Ta, Sa, Sa, t), \mathcal{M}(TTa, STa, STa, t), \mathcal{M}(Ta, STa, STa, t), \mathcal{M}(Sa, TTa, TTa, t)\}\\ &= \min\{\mathcal{M}(Ta, TTa, TTa, t), \mathcal{M}(Ta, TTa, TTa, t), \mathcal{M}(Ta, Ta, Ta, t), \mathcal{M}(TTa, TTa, TTa, t), \mathcal{M}(Ta, TTa, TTa, t), \mathcal{M}(Ta, TTa, TTa, t)\}\\ &= \min\{\mathcal{M}(Ta, TTa, TTa, t), \mathcal{M}(Ta, TTa, TTa, t), 1, 1, \mathcal{M}(Ta, TTa, TTa, t), \mathcal{M}(Ta, TTa, TTa, t)\}\\ &= \mathcal{M}(Ta, TTa, TTa, t)\end{aligned}$$

Thus, $\mathcal{M}(Ta, TTa, TTa, kt) \geq \mathcal{M}(Ta, TTa, TTa, t)$, for all $t > 0$.

Therefore by lemma 1.16, $TTa = Ta$.

Since $STa = TTa$, therefore $STa = TTa = Ta$.

Hence Ta is the common fixed point of S and T.
The proof is similar if we assume that $T(X)$ is complete subspace of X.

Uniqueness: Let u and v be two common fixed points of S and T.
Then $\mathcal{M}(u, v, v, kt) = \mathcal{M}(Tu, Tv, Tv, kt)$

$\geq \min \{\mathcal{M}(Su, Sv, Sv, t), \mathcal{M}(Tu, Tv, Tv, t), \mathcal{M}(Tu, Su, Su, t), \mathcal{M}(Tv, Sv, Sv, t),$
$\mathcal{M}(Tu, Sv, Sv, t), \mathcal{M}(Su, Tv, Tv, t)\}$
$= \min \{\mathcal{M}(u, v, v, t), \mathcal{M}(u, v, v, t), \mathcal{M}(u, u, u, t), \mathcal{M}(v, v, v, t),$
$\mathcal{M}(u, v, v, t), \mathcal{M}(u, v, v, t)\}$
$= \min \{\mathcal{M}(u, v, v, t), 1\}$
$= \mathcal{M}(u, v, v, t)$

Thus $\mathcal{M}(u, v, v, kt) \geq \mathcal{M}(u, v, v, t)$, $t > 0$.

Therefore by lemma 1.16, $u = v$.

This completes the proof.

Example: 2.18 Let $X = [0, 1]$ and $\mathcal{M}(x, y, z, t) = \frac{t}{t+|x-y|+|y-z|+|x-z|}$ for every $x, y, z \in X$ and $t > 0$. S and T be two self mappings of X defined by $Sx = \frac{2x}{5}$, $Tx = \frac{x}{5}$. Then

(1) S and T satisfy the property (E) for the sequence $x_n = \frac{1}{n}$, $n = 1, 2, 3,...$
(2) S and T are weakly compatible
(3) $T(X) \subset S(X)$
(4) For every $x, y, z \in X$ and $t > 0$, with $x \neq y$ or $y \neq z$ or $z \neq x$

$$\mathcal{M}(Tx, Ty, Tz, kt) \geq \min \{\mathcal{M}(Sx, Sy, Sz, t), \mathcal{M}(Tx, Ty, Tz, t), \mathcal{M}(Tx, Sx, Sx, t), \mathcal{M}(Ty, Sy, Sy, t), \mathcal{M}(Tx, Sy, Sz, t), \mathcal{M}(Sx, Ty, Tz, t)\}, \text{ where } 0 < k < 1.$$

Here 0 is the unique common fixed point of S and T.

Theorem: 2.19 Let S and T be two weakly compatible self mappings of a $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$ with continuous t-norm $*$ defined by $a*b = \min \{a, b\}$ for all $a, b \in [0, 1]$ such that

(i) S and T satisfy the property (E).

(ii) For every $x, y, z \in X$ and $t > 0$, with $x \neq y$ or $y \neq z$ or $z \neq x$

$$\mathcal{M}(Tx, Ty, Tz, kt) \geq \min \{\mathcal{M}(Sx, Sy, Sz, t), \mathcal{M}(Tx, Ty, Tz, t), \mathcal{M}(Tx, Sx, Sx, t)*\mathcal{M}(Ty, Sy, Sy, t), \mathcal{M}(Tx, Sy, Sz, t)*\mathcal{M}(Sx, Ty, Tz, t)\}, \text{ where } 0 < k < 1.$$

(iii) $T(X) \subset S(X)$.

(iv) $T(X)$ or $S(X)$ is complete subspace of X.

Then S and T have a unique common fixed point.

Proof: Since S and T satisfy the property (E), there exists a sequence $\{x_n\}$ in X such that
$lim_{n\to\infty} Sx_n = lim_{n\to\infty} Tx_n = x_0$, for some $x_0 \in X$.

Since $x_0 \in T(X)$ and $T(X) \subset S(X)$, there exists some point a in X such that $x_0 = Sa$, where $x_0 = lim_{n\to\infty} Sx_n$

We claim that $Ta = Sa$. Suppose that $Ta \neq Sa$.
Condition *(ii)* implies that

$$\mathcal{M}(Tx_n, Ta, Ta, kt) \geq \min \{\mathcal{M}(Sx_n, Sa, Sa, t), \mathcal{M}(Tx_n, Ta, Ta, t), \mathcal{M}(Tx_n, Sx_n, Sx_n, t)*\mathcal{M}(Ta, Sa, Sa, t), \mathcal{M}(Tx_n, Sa, Sa, t)*\mathcal{M}(Sx_n, Ta, Ta, t)\}$$

Letting $n \to \infty$ we get

$$\mathcal{M}(Sa, Ta, Ta, kt) \geq \min \{\mathcal{M}(Sa, Sa, Sa, t), \mathcal{M}(Sa, Ta, Ta, t), \mathcal{M}(Sa, Sa, Sa, t)*\mathcal{M}(Ta, Sa, Sa, t), \mathcal{M}(Sa, Sa, Sa, t)*\mathcal{M}(Sa, Ta, Ta, t)\}$$
$$= \min \{1, \mathcal{M}(Sa, Ta, Ta, t), 1*\mathcal{M}(Ta, Ta, Sa, t), 1*\mathcal{M}(Sa, Ta, Ta, t)\}$$
$$= \min \{1, \mathcal{M}(Sa, Ta, Ta, t), \mathcal{M}(Ta, Ta, Sa, t), \mathcal{M}(Sa, Ta, Ta, t)\}$$
$$= \mathcal{M}(Sa, Ta, Ta, t)$$

Thus, $\mathcal{M}(Sa, Ta, Ta, kt) \geq \mathcal{M}(Sa, Ta, Ta, t)$, $t > 0$.

Therefore, $Ta = Sa$.

Since S and T are weakly compatible, $STa = TSa = SSa = TTa$.

Now we show that Ta is common fixed point of S and T. Suppose that $Ta \neq TTa$.

$$\mathcal{M}(Ta, TTa, TTa, kt) \geq \min\{\mathcal{M}(Sa, STa, STa, t), \mathcal{M}(Ta, TTa, TTa, t), \mathcal{M}(Ta, Sa, Sa, t)*\mathcal{M}(TTa, STa, STa, t), \mathcal{M}(Ta, STa, STa, t)*\mathcal{M}(Sa, TTa, TTa, t)\}$$
$$= \min\{\mathcal{M}(Ta, TTa, TTa, t), \mathcal{M}(Ta, TTa, TTa, t), \mathcal{M}(Ta, Ta, Ta, t)*\mathcal{M}(TTa, TTa, TTa, t), \mathcal{M}(Ta, TTa, TTa, t)*\mathcal{M}(Ta, TTa, TTa, t)\}$$
$$= \min\{\mathcal{M}(Ta, TTa, TTa, t), \mathcal{M}(Ta, TTa, TTa, t), 1*1, \mathcal{M}(Ta, TTa, TTa, t)*\mathcal{M}(Ta, TTa, TTa, t)\}$$
$$= \mathcal{M}(Ta, TTa, TTa, t)$$

Thus, $\mathcal{M}(Ta, TTa, TTa, kt) \geq \mathcal{M}(Ta, TTa, TTa, t)$, for all $t > 0$.

Therefore, $TTa = Ta$.

Since $STa = TTa$, therefore $STa = TTa = Ta$.

Hence Ta is the common fixed point of S and T.

The proof is similar if we assume that $T(X)$ is complete subspace of X.

Uniqueness: Let u and v be two common fixed points of S and T.

Then $\mathcal{M}(u, v, v, kt) = \mathcal{M}(Tu, Tv, Tv, kt)$
$$\geq \min\{\mathcal{M}(Su, Sv, Sv, t), \mathcal{M}(Tu, Tv, Tv, t), \mathcal{M}(Tu, Su, Su, t)*\mathcal{M}(Tv, Sv, Sv, t), \mathcal{M}(Tu, Sv, Sv, t)*\mathcal{M}(Su, Tv, Tv, t)\}$$
$$= \min\{\mathcal{M}(u, v, v, t), \mathcal{M}(u, v, v, t), \mathcal{M}(u, u, u, t)*\mathcal{M}(v, v, v, t), \mathcal{M}(u, v, v, t)*\mathcal{M}(u, v, v, t)\}$$
$$= \min\{\mathcal{M}(u, v, v, t), \mathcal{M}(u, v, v, t), 1*1, \mathcal{M}(u, v, v, t)*\mathcal{M}(u, v, v, t)\}$$
$$= \mathcal{M}(u, v, v, t)$$

Thus $\mathcal{M}(u, v, v, kt) \geq \mathcal{M}(u, v, v, t)$, $t > 0$.

Therefore, $u = v$.

This completes the proof.

Example: 2.20 Let $X = [0, 1]$ and $a*b = \min\{a, b\}$. Let $\mathcal{M}(x, y, z, t) = \frac{t}{t+|x-y|+|y-z|+|x-z|}$ for every $x, y, z \in X$ and $t > 0$.

S and T be two self mappings of X defined by $Sx = \dfrac{x}{2}$, $Tx = \dfrac{x}{4}$. Then

(1) S and T satisfy the property (E) for the sequence $x_n = \frac{1}{n}$, $n = 1, 2, 3,...$

(2) S and T are weakly compatible

(3) $T(X) \subset S(X)$

(4) For every $x, y, z \in X$ and $t > 0$, with $x \neq y$ or $y \neq z$ or $z \neq x$

$$\mathcal{M}(Tx, Ty, Tz, kt) \geq \min\{\mathcal{M}(Sx, Sy, Sz, t), \mathcal{M}(Tx, Ty, Tz, t), \mathcal{M}(Tx, Sx, Sx, t)*\mathcal{M}(Ty, Sy, Sy, t), \mathcal{M}(Tx, Sy, Sz, t)*\mathcal{M}(Sx, Ty, Tz, t)\}, \text{ where } 0 < k < 1.$$

Here 0 is the unique common fixed point of S and T.

Theorem: 2.21 Let S and T be two weakly compatible self mappings of a $\mathcal{M}$ – fuzzy metric space $(X, \mathcal{M}, *)$ such that

(i) S and T satisfy the property (E).

(ii) For every $x, y, z \in X$ and $t > 0$, with $x \neq y$ or $y \neq z$ or $z \neq x$

$$\mathcal{M}(Tx, Ty, Tz, kt) \geq \min\{\mathcal{M}(Sx, Sy, Sz, t), \mathcal{M}(Tx, Ty, Tz, t), \tfrac{1}{2}[\mathcal{M}(Tx, Sx, Sx, t)+\mathcal{M}(Ty, Sy, Sy, t)], \tfrac{1}{2}[\mathcal{M}(Tx, Sy, Sz, t)+\mathcal{M}(Sx, Ty, Tz, t)]\}, \text{ where } 0 < k < 1.$$

(iii) $T(X) \subset S(X)$.

(iv) $T(X)$ or $S(X)$ is complete subspace of X.

Then S and T have a unique common fixed point.

Proof: Since S and T satisfy the property (E), there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} Sx_n = lim_{n\to\infty} Tx_n = x_0$, for some $x_0 \in X$.

Since $x_0 \in T(X)$ and $T(X) \subset S(X)$, there exists some point a in X such that $x_0 = Sa$, where $x_0 = lim_{n\to\infty} Sx_n$

We claim that $Ta = Sa$. Suppose that $Ta \neq Sa$.

Condition *(ii)* implies that

$$\mathcal{M}(Tx_n, Ta, Ta, kt) \geq \min\{\mathcal{M}(Sx_n, Sa, Sa, t), \mathcal{M}(Tx_n, Ta, Ta, t), ½[\mathcal{M}(Tx_n, Sx_n, Sx_n, t)+\mathcal{M}(Ta, Sa, Sa, t)], ½[\mathcal{M}(Tx_n, Sa, Sa, t)+\mathcal{M}(Sx_n, Ta, Ta, t)]\}$$

Letting $n \to \infty$ we get

$$\begin{aligned}\mathcal{M}(Sa, Ta, Ta, kt) &\geq \min\{\mathcal{M}(Sa, Sa, Sa, t), \mathcal{M}(Sa, Ta, Ta, t), ½[\mathcal{M}(Sa, Sa, Sa, t)+\mathcal{M}(Ta, Sa, Sa, t)], ½[\mathcal{M}(Sa, Sa, Sa, t)+\mathcal{M}(Sa, Ta, Ta, t)]\}\\ &= \min\{1, \mathcal{M}(Sa, Ta, Ta, t), ½[1+\mathcal{M}(Ta, Ta, Sa, t)], ½[1+\mathcal{M}(Sa, Ta, Ta, t)]\}\\ &= \min\{1, \mathcal{M}(Sa, Ta, Ta, t), ½[1+\mathcal{M}(Sa, Ta, Ta, t)], ½[1+\mathcal{M}(Sa, Ta, Ta, t)]\}\\ &= \mathcal{M}(Sa, Ta, Ta, t)\end{aligned}$$

Because, if $½[1+\mathcal{M}(Sa, Ta, Ta, t)] < \mathcal{M}(Sa, Ta, Ta, t)$, then $\mathcal{M}(Sa, Ta, Ta, t) > 1$, which is contradiction.

Thus, $\mathcal{M}(Sa, Ta, Ta, kt) \geq \mathcal{M}(Sa, Ta, Ta, t)$, for all $t > 0$.

Therefore, $Ta = Sa$.

Since S and T are weakly compatible, $STa = TSa = SSa = TTa$.

Now we show that Ta is common fixed point of S and T. Suppose that $Ta \neq TTa$.

$$\begin{aligned}\mathcal{M}(Ta, TTa, TTa, kt) &\geq \min\{\mathcal{M}(Sa, STa, STa, t), \mathcal{M}(Ta, TTa, TTa, t), ½[\mathcal{M}(Ta, Sa, Sa, t)+\mathcal{M}(TTa, STa, STa, t)], ½[\mathcal{M}(Ta, STa, STa, t)+\mathcal{M}(Sa, TTa, TTa, t)]\}\\ &= \min\{\mathcal{M}(Ta, TTa, TTa, t), \mathcal{M}(Ta, TTa, TTa, t), ½[\mathcal{M}(Ta, Ta, Ta, t)+\mathcal{M}(TTa, TTa, TTa, t)], ½[\mathcal{M}(Ta, TTa, TTa, t)+\mathcal{M}(Ta, TTa, TTa, t)]\}\\ &= \min\{\mathcal{M}(Ta, TTa, TTa, t), \mathcal{M}(Ta, TTa, TTa, t), 1, \mathcal{M}(Ta, TTa, TTa, t)]\}\\ &= \mathcal{M}(Ta, TTa, TTa, t)\end{aligned}$$

Thus, $\mathcal{M}(Ta, TTa, TTa, kt) \geq \mathcal{M}(Ta, TTa, TTa, t)$, for all $t > 0$.

Therefore, $TTa = Ta$.

Since $STa = TTa$, therefore $STa = TTa = Ta$.

Hence Ta is the common fixed point of S and T.

The proof is similar if we assume that $T(X)$ is complete subspace of X.

Uniqueness: Let u and v be two common fixed points of S and T.

Then

$$\begin{aligned}\mathcal{M}(u, v, v, kt) &= \mathcal{M}(Tu, Tv, Tv, kt)\\ &\geq \min\{\mathcal{M}(Su, Sv, Sv, t), \mathcal{M}(Tu, Tv, Tv, t), ½[\mathcal{M}(Tu, Su, Su, t)+\mathcal{M}(Tv, Sv, Sv, t)], ½[\mathcal{M}(Tu, Sv, Sv, t)+\mathcal{M}(Su, Tv, Tv, t)]\}\\ &= \min\{\mathcal{M}(u, v, v, t), \mathcal{M}(u, v, v, t), ½[\mathcal{M}(u, u, u, t)+\mathcal{M}(v, v, v, t)], ½[\mathcal{M}(u, v, v, t)+\mathcal{M}(u, v, v, t)]\}\\ &= \min\{\mathcal{M}(u, v, v, t), \mathcal{M}(u, v, v, t), 1, \mathcal{M}(u, v, v, t)\}\\ &= \mathcal{M}(u, v, v, t)\end{aligned}$$

Thus $\mathcal{M}(u, v, v, kt) \geq \mathcal{M}(u, v, v, t)$, for all $t > 0$.

Therefore, $u = v$.

This completes the proof.

Example: 2.22 Let $X = [0, 1]$ and $\mathcal{M}(x, y, z, t) = \frac{t}{t+|x-y|+|y-z|+|x-z|}$ for every $x, y, z \in X$ and $t > 0$. S and T be two self mappings of X defined by $Sx = \frac{x}{4}$, $Tx = \frac{x}{8}$. Then

(1) S and T satisfy the property (E) for the sequence $x_n = \frac{1}{n}$, $n = 1, 2, 3,...$
(2) S and T are weakly compatible
(3) $T(X) \subset S(X)$
(4) For every $x, y, z \in X$ and $t > 0$, with $x \neq y$ or $y \neq z$ or $z \neq x$

$$\mathcal{M}(Tx, Ty, Tz, kt) \geq \min \{ \mathcal{M}(Sx, Sy, Sz, t), \mathcal{M}(Tx, Ty, Tz, t), ½[\mathcal{M}(Tx, Sx, Sx, t)+\mathcal{M}(Ty, Sy, Sy, t)], ½[\mathcal{M}(Tx, Sy, Sz, t)+\mathcal{M}(Sx, Ty, Tz, t)]\}, \text{ where } 0 < k < 1.$$

Here 0 is the unique common fixed point of S and T.

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