



STRUCTURE OF SEMIRINGS AND ORDERED SEMIRINGS

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ABSTRACT

In this paper, we study some properties of semirings and ordered semirings. Also PRD semirings or semirings in which $(S, +)$ is a zeroid or C – Semiring are discussed.

Keywords: Non-negatively ordered; Non-positively ordered; PRD; Zeroid; C – semirings.

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1.INTRODUCTION:

A triple $(S, +, \cdot)$ is called a semiring if $(S, +)$ is a semigroup; (S, \cdot) is semigroup; $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for every a, b, c in S . A semiring $(S, +, \cdot)$ is said to be a totally ordered semiring if the additive semigroup $(S, +)$ and multiplicative semigroup (S, \cdot) are totally ordered semigroups under the same total order relation. An element x in a totally ordered semigroup (S, \cdot) is non - negative (non - positive) if $x^2 \geq x(x^2 \leq x)$. A totally ordered semigroup (S, \cdot) is said to be non - negatively (non - positively) ordered if every one of its elements is non-negative (non-positive). (S, \cdot) is positively (negatively) ordered in strict sense if $xy \geq x$ and $xy \geq y$ ($xy \leq x$ and $xy \leq y$) for every x and y in S . A semigroup $(S, +)$ is said to be a band if $a + a = a$ for all a in S . A semiring $(S, +, \cdot)$ is said to satisfy Integral Multiple Property (IMP) if $a^2 = na$ for all a in S where the positive integer n depends on the element a . A semiring $(S, +, \cdot)$ with additive identity zero which is multiplicative zero is said to be zero square ring if $x^2 = 0$ for all $x \in S$. In[3], the structure of semirings in which (S, \cdot) contains the multiplicative identity or (S, \cdot) is cancellative are studied. This motivated the author to study the structure of semirings in which $(S, +)$ is a zeroid or PRD semirings. Zeroid of a semiring $(S, +, \cdot)$ is the set of all x in S such that $x + y = y$ or $y + x = y$ for some y in S . We may also term this as the zeroid of $(S, +, \cdot)$. A semiring $(S, +, \cdot)$ is said to be a Positive Rational Domain (PRD) if and only if (S, \cdot) is an abelian group.

2.PASITIVE RATIONAL DOMAIN (PRD) AND ZEROID:

Theorem 2.1: Let $(S, +, \cdot)$ be a PRD and $x \in Z$, where Z is the zeroid of $(S, +, \cdot)$. Then the multiplicative identity 1 is a zeroid.

Proof: Suppose $x \in Z$. Since Z is a zeroid of $(S, +, \cdot)$, $\exists y \in S$ such that $x + y = y$ or $y + x = y$.

Suppose $x + y = y$
 $\Rightarrow x^{-1}(x + y) = x^{-1}y$
 $\Rightarrow x^{-1}x + x^{-1}y = x^{-1}y$
 $\Rightarrow 1 + x^{-1}y = x^{-1}y$
 i.e., $1 + s = s$ where $s = x^{-1}y \in S$
 $\Rightarrow 1$ is the zeroed

Suppose $y + x = y$
 $x^{-1}(y + x) = x^{-1}y$
 $\Rightarrow x^{-1}y + x^{-1}x = x^{-1}y$
 $\Rightarrow x^{-1}y + 1 = x^{-1}y$
 i.e., $p + 1 = p$ where $p = x^{-1}y \in S$
 $\Rightarrow 1$ is the zeroid
 \therefore In both cases 1 is a zeroid.

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Theorem 2.2: Let $(S, +, \bullet)$ be a PRD semiring satisfying $ab = a + b + ab$ for all $a, b \in S$. If $(S, +)$ is right cancellative, then $(S, +)$ is commutative.

Proof: By hypothesis $ab = a + b + ab$
 $ba = b + a + ba$

Since (S, \bullet) is commutative, $ab = ba$.

Therefore, $a + b + ab = b + a + ba$
 $= b + a + ab$

$\therefore a + b = b + a$, since $(S, +)$ is right cancellative

$\therefore (S, +)$ is commutative.

Theorem 2.3: If $(S, +, \bullet)$ be a PRD semiring and (S, \bullet) be a rectangular band. Then S reduces to a singleton set.

Proof: Suppose (S, \bullet) is a rectangular band

Let x be an arbitrary element of S

$$x = x(x)x$$

$$x = x^3$$

$$i.e. x^2 = x^4$$

$$\text{Also } x = x(x^2)x$$

$$\Rightarrow x = x^4$$

Therefore $x = x^2$ i.e., (S, \bullet) is a b and.

Since $(S, +, \bullet)$ is a PRD, (S, \bullet) is a group and therefore the identity is the only multiplicative idempotent in S .

Hence S reduces to a singleton set.

Examples of Zeroid Semiring 2.4:

The following are the examples of semiring in which $(S, +)$ is a zeroid. $a < b < 2a < a + b < b + a < 2b < c$

(i)

+, •	a	b	2a	a + b	b + a	2b	c
a	2a	a + b	c	c	c	c	c
b	b + a	2b	c	c	c	c	c
2a	c	c	c	c	c	c	c
.
.
.
c	c	c	c	c	c	c	c

$(S, +)$ is non-commutative. A commutative version is obtained identifying $a + b$ and $b + a$.

In example (ii) $(S, +)$ is commutative

$$a < 2a < c < b < a + b < d$$

+, •	a	2a	c	b	a + b	d
a	2a	c	c	a + b	d	d
2a	c	c	c	d	d	d
c	c	c	c	d	d	d

b	d	d	d	d	d	d
a + b	d	d	d	d	d	d
d	d	d	d	d	d	d

Theorem 2.5: Let $(S, +, \bullet)$ be a zero square semiring and 0 is the additive identity. If $(S, +)$ is a zeroid then $S^2 = \{0\}$.

Proof: Since $(S, +)$ is a zeroid

$$x + y = y \text{ or } y + x = y$$

Since S is a zero square semiring

$$x^2 = 0, y^2 = 0, \forall x, y \in S$$

$$x + y = y$$

$$\Rightarrow (x + y)y = y^2$$

$$xy + y^2 = y^2$$

$$xy + 0 = 0$$

$$xy = 0$$

$$x + y = y$$

$$\Rightarrow y(x + y) = y^2$$

$$yx + y^2 = y^2$$

$$yx + 0 = 0$$

$$yx = 0$$

If $y + x = y$ then $y(y + x) = y^2$

$$\Rightarrow y^2 + yx = y^2$$

$$\Rightarrow 0 + yx = 0$$

$$\Rightarrow yx = 0$$

Also $y + x = y$ implies $(y + x)y = y^2$

$$\Rightarrow y^2 + xy = y^2$$

$$\Rightarrow 0 + xy = 0$$

$$xy = 0$$

$$\therefore xy = yx = 0$$

Hence, $S^2 = \{0\}$

Theorem 2.6: Let $(S, +, \bullet)$ be a semiring with IMP in which $(S, +)$ is a zeroid. If $(S, +)$ is cancellative, then (S, \bullet) is a band.

Proof: Let $x \in S$. since $(S, +)$ is a zeroid

$$x + y = y \text{ or } y + x = y \text{ for some } y \text{ in } S$$

Suppose $x + y = y$

$$x + x + y = x + y$$

$$2x + y = y$$

Continuing like this, $nx + y = x + y$

$$nx = x \text{ (since } (S, +) \text{ is right cancellative)} \tag{1}$$

$$\text{since } S \text{ satisfies IMP, } x^2 = nx \tag{2}$$

∴ From (1)& (2), $x^2 = x$

If $y + x = y$
 $y + x + x = y + x$
 $y + 2x = y + x$

Continuing like this, $y + nx = y + x$

⇒ $nx = x$ (since $(S, +)$ is left cancellative) (3)

S satisfies IMP, $x^2 = nx$ (4)

∴ From (3) and (4), $x^2 = x$

∴ (S, \bullet) is a band.

3.C – SEMIRING:

In this section, we characterize c-semiring A semiring is said to be Constraint over semiring (C-semiring) if

Definition: A C – semiring is a semiring in which

- (i) $(S, +)$ is a commutative monoid
- (ii) (S, \cdot) is a commutative monoid
- (iii) $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for every a, b, c in S
- (iv) $a \cdot 0 = 0 \cdot a = 0$
- (v) $(S, +)$ is a band and 1 is the absorbing element of '+'

Example 3.1: Let S be a C – semiring. Define $nx + 0 = nx = 0 + nx$, $x + 1 = 1 = 1 + x$, $\forall x \in S$, then (S, \cdot) is not necessarily band.

+	1	x	y	0
1	1	1	1	1
x	1	x	y	x
y	1	y	y	y
0	1	x	y	0

•	1	x	y	0
1	1	x	y	0
x	x	0	0	0
y	y	0	0	0
0	0	0	0	0

Theorem 3.2: Let $(S, +, \bullet)$ be a C – semiring and $(S, +)$ be left cancellative or right cancellative then (S, \bullet) is a band.

Proof: Let $a \in S$

$$\begin{aligned} \text{Consider } a + a^2 &= a \cdot 1 + a^2 \\ &= a(1 + a) \\ &= a \cdot 1 \\ &= a \text{ --- (I)} \end{aligned}$$

Also $a + a = a \text{ --- (II)}$

From (I) and (II), $a + a^2 = a + a$

⇒ $a^2 = a$ (since $(S, +)$ is left cancellative)

∴ (S, \bullet) is a band

Similarly if $(S, +)$ is right cancellative, we can prove that (S, \cdot) is band.

Theorem 3.3: Let $(S, +, \bullet)$ be a C – semiring, then $a^2 = a + a^2$, $\forall a \in S$ if and only if (S, \bullet) is a band.

Proof: Suppose $a^2 = a + a^2$

$$\begin{aligned} &= a \cdot 1 + a^2 \\ &= a(1 + a) \\ &= a \cdot 1 \text{ (Since 1 is an absorbing element w.r.to.'+') } \\ a^2 &= a \end{aligned}$$

$\therefore (S, \bullet)$ is a band

Conversely suppose (S, \bullet) is a band

$$\begin{aligned} \text{i. e., } a^2 &= a, \forall a \in S \\ a^2 &= a \cdot 1 \\ &= a(1 + a) \text{ (Since 1 is an absorbing element w.r to. ' + ')} \\ &= a + a^2, \forall a \in S \\ \therefore a^2 &= a + a^2 \end{aligned}$$

Theorem 3.4: Let $(S, +, \bullet)$ be a C – semiring. Then (i) $(S, +)$ is regular. (ii) $(S, +)$ is Zeroid.

Proof: (i) Let $x \in S$

$$\begin{aligned} \text{Then } x + 0 + x &= x + x \quad (\because 0 \text{ is the additive identity}) \\ &= x \quad (\because (S, +) \text{ is a band}) \end{aligned}$$

$\therefore S$ is regular.

(ii) Let $x \in S$

$$\text{Then } x + 1 = 1, \forall x \in S \quad (\because S \text{ is a C – Semiring})$$

$\therefore (S, +)$ is a zeroid.

Theorem 3.5: Let $(S, +, \bullet)$ be a C – semiring. Then S contains two elements a and b such that $ab = a + b + ab$ if and only if $ab = a + b = a = b$

Proof: $ab = a + b + ab$
 $= a + (1 + a)b$
 $ab = a + b \quad \text{----- (I)}$
 $ab = a + b + ab$
 $ab + b = a + b + ab + b$
 $(a + 1)b = a + b + (a + 1)b$
 $1 \cdot b = a + b + 1 \cdot b$

$$\begin{aligned} \Rightarrow 1 \cdot b &= a + b + b \\ b &= a + b \quad \text{----- (II) (since } b + b = b) \\ ab &= a + b + ab \end{aligned}$$

$$\begin{aligned} \Rightarrow ab + a &= a + b + ab + a \\ a(b + 1) &= a + b + a(b + 1) \\ a \cdot 1 &= a + b + a \cdot 1 \\ a &= a + b \quad \text{----- (III)} \end{aligned}$$

from (I), (II) and (III), $ab = a + b = a = b$

conversely, $ab = a + b$
 $= a + b + b$ (since $b + b = b$)
 $= a + b + ab$
 $\therefore ab = a + b + ab$

Examples 3.6:

The following are the examples of ordered C – semirings.

(i)

+	I	x	0
I	I	I	I
x	I	x	x
0	I	x	0

•	I	x	0
I	I	x	0
x	x	0	0
0	0	0	0

(ii)

+	I	x	0
I	I	I	I
x	I	x	x
0	I	x	0

•	I	x	0
I	I	x	0
x	x	x	0
0	0	0	0

(i) $(S, +)$ is p.t.o. and $0 < x < 1$.

+	0	x	1
0	0	x	1
x	x	x	1
1	1	1	1

•	0	x	1
0	0	0	0
x	0	0	x
1	0	x	1

Theorem 3.7: Let $(S, +, \bullet)$ be a t.o. C – semiring in which $(S, +)$ is p.t.o.(n.t.o.), then for any x, y in S , $x + y = x$ or y .

Proof: Suppose $x < y$, Then $x + y \leq y + y$

$$x + y \leq y \text{ --- (I) (since } y + y = y)$$

By hypothesis $(S, +)$ is p.t.o.,

$$\text{i.e., } x + y \geq y \text{ --- (II)}$$

from (I)&(II) $x + y = y$

If $y < x$

$$x + y \leq x + x$$

$$x + y \leq x$$

Since $(S, +)$ is p.t.o.

$$x + y \geq x$$

$$\therefore x + y = x$$

Theorem 3.8: Let $(S, +, \bullet)$ be a t.o. C – semiring in which (S, \bullet) is p.t.o. Then 1 is the minimum element and 0 is the maximum element.

Proof: $a \cdot 0 = 0 \cdot a = 0$ since 0 is the multiplicative zero

Since (S, \bullet) is p.t.o. 0 is the maximum element

$$a = a \cdot 1 \geq 0$$

$\therefore 1$ is the minimum element.

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