

On α - Generalized Star Regular Closed Sets in Bitopological Spaces

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ABSTRACT

The aim of this paper is to introduce the concept of αg^*r -closed sets in bitopological spaces and the newly related concept of pairwise αg^*r -continuous mappings. Also αG^*RO -connectedness and αG^*RO -compactness are introduced in bitopological spaces and some of their properties are established.

Keywords: αg^*r -closed sets, pairwise αg^*r -continuous, αG^*RO -connectedness, αG^*RO -compactness.

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1 INTRODUCTION

The concept of bitopological spaces was introduced by Kelly [4] in 1963. Separation axioms in bitopological spaces were first studied by him. Fukutake [3] introduced generalized closed sets and pairwise generalized closure operator in bitopological spaces in 1986. On the other hand Chandrasekhara Rao and Kannan [1] introduced the concept of generalized star regular closed sets in bitopological spaces. Vadivel, Vijayalakshmi and Krishnaswamy [7] introduced the concepts of α -generalized star closed sets in bitopological spaces. The connectedness and components were introduced by Pervin [5] in bitopological spaces. A detailed study of connectedness in bitopological spaces was carried out by Reilly [6]. Fletcher, Hoyle III, and Patty introduced the notion of a pairwise compact bitopological spaces and proved that every pairwise Hausdorff pairwise compact space is pairwise regular.

In this paper, we introduce the concept of αg^*r -closed sets in bitopological spaces and the newly related concept of pairwise αg^*r -continuous mappings. Also αG^*RO -connectedness and αG^*RO -compactness are introduced in bitopological spaces and some of their properties are established.

2 PRELIMINARIES

Throughout this paper we shall denote by (X, τ_1, τ_2) a bitopological space. For any subset $A \subseteq X$, $\tau_i\text{-int}(A)$ and $\tau_i\text{-cl}(A)$ denote the interior of A and the closure of A with respect to τ_i for $i = 1, 2$.

We shall require the following known definitions:

Definition 2.1: Let (X, τ) be a topological spaces. A subset A of X is called

- a regular closed set if $A = \text{cl}(\text{int}(A))$.
- an α -open set if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$.

Let (X, τ_1, τ_2) or simply X denote a bitopological space. For any subset $A \subseteq X$, the intersection (resp. union) of all τ_i -regular closed sets containing A (resp. τ_i -regular open sets contained in A) is called the τ_i -regular closure (resp. τ_i -regular interior) of A , denoted by $\tau_i\text{-rcl}(A)$ (resp. $\tau_i\text{-rint}(A)$), $i = 1, 2$. The regular closure and regular interior of B relative to A with respect to the topology τ_i are written as $\tau_i\text{-rcl}_A(B)$ and $\tau_i\text{-rint}_A(B)$ for $i = 1, 2$, respectively. The set of all τ_i -regular closed (resp. τ_i -regular open) sets in X is denoted by $\tau_i\text{-RC}(X, \tau_1, \tau_2)$, (resp. $\tau_i\text{-RO}(X, \tau_1, \tau_2)$), $i = 1, 2$. The set of all τ_i - α -open sets in X is denoted by $\tau_i\text{-}\alpha\text{O}(X, \tau_1, \tau_2)$.

Definition 2.2: Let (X, τ_1, τ_2) be a bitopological space. A subset A of X is called

- $\tau_1\tau_2$ - α generalized star closed (briefly, $\tau_1\tau_2$ - αg^* closed) in X if $\tau_2\text{-cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1 - α -open in X .

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- $\tau_1\tau_2$ - α generalized star open (briefly, $\tau_1\tau_2$ - αg^* open) in X if $U \subseteq \tau_2\text{-int}(A)$ whenever $U \subseteq A$ and U is τ_1 - α -closed in X .
- $\tau_1\tau_2$ -generalized star regular closed (briefly, $\tau_1\tau_2$ - g^*r closed) in X if $\tau_2\text{-rcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1 -open in X .
- $\tau_1\tau_2$ -generalized star regular open (briefly, $\tau_1\tau_2$ - g^*r open) in X if $U \subseteq \tau_2\text{-rint}(A)$ whenever $U \subseteq A$ and U is τ_1 -closed in X .

Definition 2.3: Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise continuous if $f : (X, \tau_1) \rightarrow (Y, \sigma_1)$ and $f : (X, \tau_2) \rightarrow (Y, \sigma_2)$ are continuous.

3 $\tau_1\tau_2$ - α generalized star regular closed sets

Definition 3.1: A set A of bitopological space (X, τ_1, τ_2) is called τ_i - α generalized star regular closed (briefly, τ_i - αg^*r closed) if $\tau_i\text{-rcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_i - α -open in X , $i = 1, 2$. The complement of a τ_i - αg^*r closed set is said to be τ_i - αg^*r open.

For any subset $A \subseteq X$, the intersection (resp. union) of all τ_i - αg^*r closed sets containing A (resp. τ_i - αg^*r open sets contained in A) is called the τ_i - αg^*r closure (resp. τ_i - αg^*r interior) of A , denoted by $\tau_i\text{-}\alpha g^*r\text{-cl}(A)$ (resp. $\tau_i\text{-}\alpha g^*r\text{-int}(A)$), $i = 1, 2$. The set of all τ_i - αg^*r closed (resp. τ_i - αg^*r open) sets in X is denoted by $\tau_i\text{-}\alpha G^*RC(X, \tau_1, \tau_2)$ (resp. $\tau_i\text{-}\alpha G^*RO(X, \tau_1, \tau_2)$), $i = 1, 2$.

Theorem 3.2: Every τ_i -regular closed set is τ_i - αg^*r closed.

Definition 3.3: A set A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$ - α generalized star regular closed (briefly, $\tau_1\tau_2$ - αg^*r closed) if $\tau_2\text{-rcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1 - α -open in X . The complement of a $\tau_1\tau_2$ - αg^*r closed set is said to be $\tau_1\tau_2$ - αg^*r open. The set of all $\tau_1\tau_2$ - αg^*r closed sets in X is denoted by $\tau_1\tau_2\text{-}\alpha G^*RC(X)$ and the set of all $\tau_1\tau_2$ - αg^*r open sets in X is denoted by $\tau_1\tau_2\text{-}\alpha G^*RO(X)$.

Example 3.4: Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{b\}\}$. Then $\emptyset, X, \{b, c\}$ are $\tau_1\tau_2$ - αg^*r closed and $\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}$ are not $\tau_1\tau_2$ - αg^*r closed.

Theorem 3.5: Every τ_2 -regular closed set is $\tau_1\tau_2$ - αg^*r closed.

Proof: Let A be a τ_2 -regular closed set. Let $A \subseteq U$ and U be τ_1 - α -open in X . Since A is τ_2 -regular closed in X , we have $\tau_2\text{-rcl}(A) = A \subseteq U$. Therefore, A is $\tau_1\tau_2$ - αg^*r closed set.

Theorem 3.6: If A and B are $\tau_1\tau_2$ - αg^*r closed sets then $A \cup B$ is $\tau_1\tau_2$ - αg^*r closed.

Proof: Suppose that A and B are $\tau_1\tau_2$ - αg^*r closed. Let U be τ_1 - α -open and $A \cup B \subseteq U$. Since $A \cup B \subseteq U$, we have $A \subseteq U$ and $B \subseteq U$. Since A and B are $\tau_1\tau_2$ - αg^*r closed sets, we have $\tau_2\text{-rcl}(A) \subseteq U$ and $\tau_2\text{-rcl}(B) \subseteq U$. Therefore, $[\tau_2\text{-rcl}(A)] \cup [\tau_2\text{-rcl}(B)] \subseteq U$. Since $[\tau_2\text{-rcl}(A)] \cup [\tau_2\text{-rcl}(B)] = \tau_2\text{-rcl}(A \cup B)$, we have $\tau_2\text{-rcl}(A \cup B) \subseteq U$. Hence $A \cup B$ is $\tau_1\tau_2$ - αg^*r closed.

Remark 3.7: The converse of the above theorem is not true in general as can be seen from the following example.

Example 3.8: Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{a\}$ and $B = \{b\}$. Then $A \cup B = \{a, b\}$ is $\tau_1\tau_2$ - αg^*r closed. But $A = \{a\}$ is not $\tau_1\tau_2$ - αg^*r closed.

Remark 3.9: Intersection of two $\tau_1\tau_2$ - αg^*r closed sets need not be $\tau_1\tau_2$ - αg^*r closed as can be seen from the following example.

Example 3.10: Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{a, b\}$ and $B = \{a, c\}$. Then $A \cap B = \{a\}$ is not $\tau_1\tau_2$ - αg^*r closed. But $A = \{a, b\}$ and $B = \{a, c\}$ are $\tau_1\tau_2$ - αg^*r closed.

Definition 3.11: Let (X, τ_1, τ_2) be a bitopological space. A collection of subsets of X is said to be τ_i -regular locally finite, if for each point x in X there is a τ_i -regular open set U containing x such that U intersects only finitely many of the sets in the collection.

Theorem 3.12: If $\{A_i, i \in I\}$ is a τ_i -regular locally finite family, then $\tau_i\text{-rcl}[\cup(A_i)] = \cup\tau_i\text{-rcl}(A_i)$.

Theorem 3.13: The arbitrary union of $\tau_1\tau_2$ - αg^*r closed sets $A_i, i \in I$ in a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$ - αg^*r closed if the family $\{A_i, i \in I\}$ is τ_2 -regular locally finite.

Proof: Let $\{A_i, i \in I\}$ be τ_2 -regular locally finite and A_i be $\tau_1\tau_2$ - αg^*r closed in X for each $i \in I$. Let $\cup A_i \subseteq U$ and U be τ_1 - α -open in X . Then $A_i \subseteq U$ and U is τ_1 - α -open in X for each i . Since A_i is $\tau_1\tau_2$ - αg^*r closed in X for each $i \in I$, we have $\tau_2\text{-rcl}(A_i) \subseteq U$. Consequently, $\cup[\tau_2\text{-rcl}(A_i)] \subseteq U$.

Since the family $\{A_i, i \in I\}$ is τ_2 -regular locally finite, by Theorem 3.12, $\tau_2\text{-rcl}[\cup(A_i)] = \cup[\tau_2\text{-rcl}(A_i)] \subseteq U$. Therefore, $\cup A_i$ is $\tau_1\tau_2$ - αg^*r closed in X .

Theorem 3.14: The arbitrary intersection of $\tau_1\tau_2$ - αg^*r open sets $A_i, i \in I$ in a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$ - αg^*r open if the family $\{A_i^c, i \in I\}$ is τ_2 -regular locally finite.

Proof: Let $\{A_i^c, i \in I\}$ be τ_2 -regular locally finite and A_i be $\tau_1\tau_2$ - αg^*r open in X , for each $i \in I$. Then A_i^c is $\tau_1\tau_2$ - αg^*r closed in X , for each $i \in I$. Hence by Theorem 3.13, we have $\cup(A_i^c)$ is $\tau_1\tau_2$ - αg^*r closed in X . Consequently, $(\cap(A_i))^c$ is $\tau_1\tau_2$ - αg^*r closed in X . Therefore, $\cap A_i$ is $\tau_1\tau_2$ - αg^*r open in X .

Theorem 3.15: Let A be a subset of a bitopological space (X, τ_1, τ_2) . If A is $\tau_1\tau_2$ - αg^*r closed then $\tau_2\text{-rcl}(A) - A$ contains no nonempty τ_1 - α -closed set.

Proof: Suppose that A is $\tau_1\tau_2$ - αg^*r closed. Let F be a τ_1 - α -closed set such that $F \subseteq \tau_2\text{-rcl}(A) - A$. We shall show that $F = \emptyset$. Since $F \subseteq \tau_2\text{-rcl}(A) - A$, we have $A \subseteq F^c$ and $F \subseteq \tau_2\text{-rcl}(A)$. Since F is a τ_1 - α -closed set, we have F^c is a τ_1 - α -open. Since A is $\tau_1\tau_2$ - αg^*r closed, we have $\tau_2\text{-rcl}(A) \subseteq F^c$.

Thus $F \subseteq [\tau_2\text{-rcl}(A)]^c = X - [\tau_2\text{-rcl}(A)]$. Hence $F \subseteq [\tau_2\text{-rcl}(A)] \cap [X - [\tau_2\text{-rcl}(A)]] = \emptyset$. Therefore, $F = \emptyset$. Hence $\tau_2\text{-rcl}(A) - A$ contains no nonempty τ_1 - α -closed sets.

Theorem 3.16: Let A be a $\tau_1\tau_2$ - αg^*r closed set. Then A is τ_2 -closed in X if and only if $\tau_2\text{-cl}(A) - A$ is τ_1 - α -closed in X .

Proof: Suppose that A is $\tau_1\tau_2$ - αg^*r closed. Let A be τ_2 -closed. Then $\tau_2\text{-cl}(A) = A$. Therefore, $\tau_2\text{-cl}(A) - A = \emptyset$ is τ_1 - α -closed.

Conversely, suppose that A is $\tau_1\tau_2$ - αg^*r closed and $\tau_2\text{-cl}(A) - A$ is τ_1 - α -closed. Since $\tau_2\text{-cl}(A) \subseteq \tau_2\text{-rcl}(A)$, we have $\tau_2\text{-cl}(A) - A \subseteq \tau_2\text{-rcl}(A) - A$, for any subset A of X . Since A is $\tau_1\tau_2$ - αg^*r closed, we have $\tau_2\text{-cl}(A) - A = \emptyset$, by Theorem 3.15. Hence A is τ_2 -closed.

Theorem 3.17: Let A and B be subsets such that $A \subseteq B \subseteq \tau_2\text{-rcl}(A)$. If A is $\tau_1\tau_2$ - αg^*r closed then B is $\tau_1\tau_2$ - αg^*r closed.

Proof: Let A and B be subsets such that $A \subseteq B \subseteq \tau_2\text{-rcl}(A)$. Suppose that A is $\tau_1\tau_2$ - αg^*r closed. Let $B \subseteq U$ and U be τ_1 - α -open in X . Since $A \subseteq B$ and $B \subseteq U$, we have $A \subseteq U$. Since A is $\tau_1\tau_2$ - αg^*r closed, we have $\tau_2\text{-rcl}(A) \subseteq U$. Since $B \subseteq \tau_2\text{-rcl}(A)$, we have $\tau_2\text{-rcl}(B) \subseteq \tau_2\text{-rcl}[\tau_2\text{-rcl}(A)] = \tau_2\text{-rcl}(A) \subseteq U$. Therefore B is $\tau_1\tau_2$ - αg^*r closed.

Theorem 3.18: If A is $\tau_1\tau_2$ - αg^*r closed and $A \subseteq B \subseteq \tau_2\text{-rcl}(A)$ then $\tau_2\text{-rcl}(B) - B$ contains no nonempty τ_1 - α -closed set.

Proof: Follows from Theorem 3.17 and 3.15.

Theorem 3.19: Suppose that $\tau_1\text{-}\alpha O(X, \tau_1, \tau_2) \subseteq \tau_2\text{-RC}(X, \tau_1, \tau_2)$. Then every subset of X is $\tau_1\tau_2$ - αg^*r closed.

Proof. Let A be a subset of X . Let $A \subseteq U$ and U be τ_1 - α -open in X . Since $\tau_1\text{-}\alpha O(X, \tau_1, \tau_2) \subseteq \tau_2\text{-RC}(X, \tau_1, \tau_2)$, we have U is τ_2 -regular closed in X . Therefore, $\tau_2\text{-rcl}(U) = U$. Since $A \subseteq U$, we have $\tau_2\text{-rcl}(A) \subseteq \tau_2\text{-rcl}(U) = U$. Therefore, A is $\tau_1\tau_2$ - αg^*r closed.

Theorem 3.20: Let $B \subseteq A$ where A is τ_1 - α -open and $\tau_1\tau_2$ - αg^*r closed. Then B is $\tau_1\tau_2$ - αg^*r closed relative to A if and only if B is $\tau_1\tau_2$ - αg^*r closed in X .

Proof: Let $B \subseteq A$ where A is τ_1 - α -open and $\tau_1\tau_2$ - αg^*r closed. Suppose that B is $\tau_1\tau_2$ - αg^*r closed relative to A . We shall show that B is $\tau_1\tau_2$ - αg^*r closed in X . Let $B \subseteq U$ and U is τ_1 - α -open in X . Since A and U are τ_1 - α -open sets in X , we have $A \cap U$ is τ_1 - α -open in A . Since $B \subseteq U$ and $B \subseteq A$, we have $B \subseteq U \cap A$. Since B is $\tau_1\tau_2$ - αg^*r closed relative to A , we have $\tau_2\text{-rcl}_A(B) \subseteq A \cap U$. Since $A \subseteq A$ and A is τ_1 - α -open in X , we have $\tau_2\text{-rcl}(A) \subseteq A$ (since A is $\tau_1\tau_2$ - αg^*r

closed in X). Since $B \subseteq A$, $\tau_2\text{-rcl}(B) \subseteq \tau_2\text{-rcl}(A)$. Hence $\tau_2\text{-rcl}(B) \subseteq A$. Therefore, $\tau_2\text{-rcl}(B) \cap A = \tau_2\text{-rcl}(B) \Rightarrow \tau_2\text{-rcl}_A(B) = \tau_2\text{-rcl}(B)$. Hence $\tau_2\text{-rcl}(B) \subseteq A \cap U$. Thus B is $\tau_1\tau_2\text{-}\alpha g^*r$ closed.

Conversely, suppose that B is $\tau_1\tau_2\text{-}\alpha g^*r$ closed in X . We shall show that B is $\tau_1\tau_2\text{-}\alpha g^*r$ closed relative to A . Let $B \subseteq U$ and U be $\tau_1\text{-}\alpha$ -open in A . Since U is $\tau_1\text{-}\alpha$ -open in A , we have $U = V \cap A$, where V is $\tau_1\text{-}\alpha$ -open in X .

Hence $B \subseteq U \subseteq V$. Since B is $\tau_1\tau_2\text{-}\alpha g^*r$ closed in X , $\tau_2\text{-rcl}(B) \subseteq V$. Hence $\tau_2\text{-rcl}(B) \cap A \subseteq V \cap A$, which in turn implies that $\tau_2\text{-rcl}_A(B) \subseteq V \cap A = U$. Therefore B is $\tau_1\tau_2\text{-}\alpha g^*r$ closed relative to A .

Theorem 3.21: For each $x \in X$, the singleton $\{x\}$ is either $\tau_1\text{-}\alpha$ -closed or $\tau_1\tau_2\text{-}\alpha g^*r$ open.

Proof: Let $x \in X$ and suppose that $\{x\}$ is not $\tau_1\text{-}\alpha$ -closed. Then $X - \{x\}$ is not $\tau_1\text{-}\alpha$ -open. Consequently, X is the only $\tau_1\text{-}\alpha$ -open set containing the set $X - \{x\}$. Therefore $X - \{x\}$ is $\tau_1\tau_2\text{-}\alpha g^*r$ closed. Hence $\{x\}$ is $\tau_1\tau_2\text{-}\alpha g^*r$ open.

4 Pairwise αg^*r -continuous Functions

Definition 4.1: A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise αg^*r -continuous if $f^{-1}(U)$ is $\tau_i\tau_j\text{-}\alpha g^*r$ closed in X for each σ_j -closed set U in Y , $i \neq j$ and $i, j = 1, 2$.

Example 4.2: Let $X = \{a, b, c\}$ and $Y = \{p, q, r\}$. Consider the topologies $\tau_1 = \{\emptyset, X, \{a\}, \{b, c\}\}$, $\tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\sigma_1 = \{\emptyset, Y, \{p\}\}$ and $\sigma_2 = \{\emptyset, Y, \{q\}, \{p\}, \{p, q\}\}$. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function defined by $f(a) = q$, $f(b) = r$, $f(c) = p$. Then f is pairwise αg^*r -continuous.

Theorem 4.3: The following are equivalent for a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$

- (a) f is pairwise αg^*r -continuous
- (b) $f^{-1}(A)$ is $\tau_i\tau_j\text{-}\alpha g^*r$ open for each σ_j -open set A in Y , $i \neq j$ and $i, j = 1, 2$.

Proof: (a) \Rightarrow (b)

Suppose that f is pairwise αg^*r -continuous. Let A be σ_j -open in Y . Then A^c is σ_j -closed in Y . Since f is pairwise αg^*r -continuous, we have $f^{-1}(A^c)$ is $\tau_i\tau_j\text{-}\alpha g^*r$ closed in X , $i \neq j$ and $i, j = 1, 2$. Consequently, $f^{-1}(A)$ is $\tau_i\tau_j\text{-}\alpha g^*r$ open in X , $i \neq j$ and $i, j = 1, 2$.

(b) \Rightarrow (a)

Suppose that $f^{-1}(A)$ is $\tau_i\tau_j\text{-}\alpha g^*r$ open for each σ_j -open set A in Y , $i \neq j$ and $i, j = 1, 2$. Let V be σ_j -closed in Y . Then V^c is σ_j -open in Y . Therefore, by our assumption, $f^{-1}(V^c)$ is $\tau_i\tau_j\text{-}\alpha g^*r$ open in X , $i \neq j$ and $i, j = 1, 2$. Hence $f^{-1}(V)$ is $\tau_i\tau_j\text{-}\alpha g^*r$ closed in X , $i \neq j$ and $i, j = 1, 2$. Therefore f is pairwise αg^*r -continuous.

Definition 4.4: Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise regular continuous if $f^{-1}(U)$ is τ_i -regular closed in X for each σ_i -closed set U in Y , $i = 1, 2$.

Theorem 4.5: Every pairwise regular continuous function is pairwise αg^*r -continuous.

Proof: Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be pairwise regular continuous. Let U be a σ_j -closed set in Y . Then $f^{-1}(U)$ is τ_j -regular closed in X . Since every τ_j -regular closed set is $\tau_i\tau_j\text{-}\alpha g^*r$ closed, $i \neq j$ and $i, j = 1, 2$, we have f is pairwise αg^*r -continuous.

Definition 4.6: A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise αg^*r -irresolute if $f^{-1}(U)$ is $\tau_i\tau_j\text{-}\alpha g^*r$ closed for each $\sigma_i\sigma_j\text{-}\alpha g^*r$ closed set in Y , $i \neq j$ and $i, j = 1, 2$.

Example 4.7: Let $X = \{a, b, c\}$ and $Y = \{p, q, r\}$. Consider the topologies $\tau_1 = \{\emptyset, X, \{a\}, \{b, c\}\}$, $\tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\sigma_1 = \{\emptyset, Y, \{p\}\}$, $\sigma_2 = \{\emptyset, Y, \{p\}, \{q\}, \{p, q\}\}$. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function defined by $f(a) = q$, $f(b) = r$, $f(c) = p$. Then f is pairwise αg^*r -irresolute.

Concerning the composition of functions, we have the following.

Theorem 4.8: Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2)$ be two functions.

- (1) If f and g are pairwise αg^*r -irresolute, then $g \circ f$ is also pairwise αg^*r -irresolute.
- (2) If f is pairwise αg^*r -irresolute and g is pairwise αg^*r -continuous then $g \circ f$ is pairwise αg^*r -continuous.
- (3) If f is pairwise αg^*r -continuous and g is pairwise continuous then $g \circ f$ is pairwise αg^*r -continuous.

Proof: (1) Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2)$ be pairwise αg^*r -irresolute. Let U be $\mu_i\mu_j$ - αg^*r closed in Z , $i \neq j$ and $i, j = 1, 2$. Since g is pairwise αg^*r -irresolute, $g^{-1}(U)$ is $\sigma_i\sigma_j$ - αg^*r closed in Y , $i \neq j$ and $i, j = 1, 2$. Since f is pairwise αg^*r -irresolute, $(g \circ f)^{-1} = f^{-1}(g^{-1}(U))$ is $\tau_i\tau_j$ - αg^*r closed in X , $i \neq j$ and $i, j = 1, 2$. Therefore, $g \circ f$ is pairwise αg^*r -irresolute.

(2) Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be pairwise αg^*r -irresolute and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2)$ be pairwise αg^*r -continuous. Let U be μ_j -closed in Z . Since g is pairwise αg^*r -continuous, $g^{-1}(U)$ is $\sigma_i\sigma_j$ - αg^*r closed set in Y , $i \neq j$ and $i, j = 1, 2$. Since f is pairwise αg^*r -irresolute, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is $\tau_i\tau_j$ - αg^*r closed in X , $i \neq j$ and $i, j = 1, 2$. Therefore $g \circ f$ is pairwise αg^*r -continuous.

(3) Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be pairwise αg^*r -continuous and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2)$ be pairwise continuous. Let U be μ_j -closed in Z . Since g is pairwise continuous, $g^{-1}(U)$ is σ_j -closed in Y . Since f is pairwise αg^*r -continuous, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is $\tau_i\tau_j$ - αg^*r closed in X , $i \neq j$ and $i, j = 1, 2$. Therefore $g \circ f$ is pairwise αg^*r -continuous.

Remark 4.9: The composition of two pairwise αg^*r -continuous functions need not be a pairwise αg^*r -continuous function as can be seen from the following example:

Example 4.10: Let $X = \{p, q, r\}$, $Y = \{a, b, c\}$ and $Z = \{s, t, u\}$. Consider $\tau_1 = \{\emptyset, X, \{p\}\}$, $\tau_2 = \{\emptyset, X, \{p\}, \{q\}, \{p, q\}\}$, $\sigma_1 = \{\emptyset, Y, \{a\}, \{b, c\}\}$, $\sigma_2 = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$, $\mu_1 = \{\emptyset, Z, \{u\}\}$ and $\mu_2 = \{\emptyset, Z, \{u\}, \{t\}, \{u, t\}\}$. Then $\tau_1\tau_2$ - $\alpha G^*RC(X) = \{\emptyset, X, \{r\}, \{p, r\}, \{q, r\}\}$ and $\sigma_1\sigma_2$ - $\alpha G^*RC(Y) = \{\emptyset, Y, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function defined by $f(p) = h$, $f(q) = a$, $f(r) = c$ and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \mu_1, \mu_2)$ be a function defined by $g(a) = t$, $g(b) = s$, $g(c) = u$. Then f and g are αg^*r -pairwise continuous. But $(g \circ f)^{-1}(\{s, t\}) = \{p, q\}$. Since $\{a, b\} \in \sigma_1\sigma_2$ - $\alpha G^*RC(Y)$ and $\{p, q\} \notin \tau_1\tau_2$ - $\alpha G^*RC(X)$, we get $g \circ f$ is not αg^*r -pairwise continuous.

5 Pairwise αG^*RO -connected spaces

Definition 5.1: A bitopological space (X, τ_1, τ_2) is *pairwise αG^*RO -connected* if X cannot be expressed as the union of two non empty disjoint sets A and B such that $[A \cap \tau_1\text{-}\alpha g^*r\text{cl}(B)] \cup [\tau_2\text{-}\alpha g^*r\text{cl}(A) \cap B] = \emptyset$.

Suppose X can be so expressed then X is called *pairwise αG^*RO -disconnected* and we write $X = A|B$ and call this pairwise αG^*RO -separation of X .

Example 5.2: Let $X = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\tau_2 = \{\emptyset, X, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then (X, τ_1, τ_2) is pairwise αG^*RO -connected.

Example 5.3: Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ_1, τ_2) is pairwise αG^*RO -disconnected, since $X = A|B$ gives a pairwise αG^*RO -separation of X .

Theorem 5.4: The following conditions are equivalent for any bitopological space:

- (a) X is pairwise αG^*RO -connected.
- (b) X cannot be expressed as the union of two nonempty disjoint sets A and B such that A is τ_1 - αg^*r open and B is τ_2 - αg^*r open
- (c) X contains no nonempty proper subset which is both τ_1 - αg^*r open and τ_2 - αg^*r closed.

Proof: (a) \Rightarrow (b)

Assume that X is pairwise αG^*RO -connected. Suppose that X can be expressed as the union of two nonempty disjoint sets A and B such that A is τ_1 - αg^*r open and B is τ_2 - αg^*r open. Then $A \cap B = \emptyset$. Consequently $A \subseteq B^c$. Then τ_2 - $\alpha g^*r\text{cl}(A) \subseteq \tau_2$ - $\alpha g^*r\text{cl}(B^c) = B^c$. Therefore, τ_2 - $\alpha g^*r\text{cl}(A) \cap B = \emptyset$. Similarly we can prove $A \cap \tau_1$ - $\alpha g^*r\text{cl}(B) = \emptyset$. Hence $[A \cap \tau_1$ - $\alpha g^*r\text{cl}(B)] \cup [\tau_2$ - $\alpha g^*r\text{cl}(A) \cap B] = \emptyset$. This is a contradiction. Hence X cannot be expressed as the union of two nonempty disjoint sets A and B such that A is τ_1 - αg^*r open and B is τ_2 - αg^*r open.

(b) \Rightarrow (c)

Assume that X cannot be expressed as the union of two nonempty disjoint sets A and B such that A is τ_1 - αg^*r open and B is τ_2 - αg^*r open. Suppose that X contains a nonempty proper subset A which is both τ_1 - αg^*r open and τ_2 - αg^*r closed. Then $X = A \cup A^c$, where A is τ_1 - αg^*r open, A^c is τ_2 - αg^*r open and A and A^c are disjoint. This is a contradiction to our assumption. Therefore, X contains no nonempty proper subset which is both τ_1 - αg^*r open and τ_2 - αg^*r closed.

(c) \Rightarrow (d)

Assume that X contains no nonempty proper subset which is both τ_1 - αg^*r open and τ_2 - αg^*r closed. Suppose that X is pairwise αG^*RO -disconnected. Then X can be expressed as the union of two nonempty disjoint sets A and B such that $[A \cap \tau_1$ - $\alpha g^*r\text{cl}(B)] \cup [\tau_2$ - $\alpha g^*r\text{cl}(A) \cap B] = \emptyset$. Since $A \cap B = \emptyset$, we have $A = B^c$ and $B = A^c$.

Since $\tau_2\text{-}\alpha\text{g}^*\text{rcl}(A) \cap B = \emptyset$, we have $\tau_2\text{-}\alpha\text{g}^*\text{rcl}(A) \subseteq B^c$. Hence $\tau_2\text{-}\alpha\text{g}^*\text{rcl}(A) \subseteq A$. Therefore, A is $\tau_2\text{-}\alpha\text{g}^*\text{r}$ closed. Similarly, B is $\tau_1\text{-}\alpha\text{g}^*\text{r}$ closed.

Since $A = B^c$, A is $\tau_1\text{-}\alpha\text{g}^*\text{r}$ open. Therefore, there exists a nonempty proper set A which is both $\tau_1\text{-}\alpha\text{g}^*\text{r}$ open and $\tau_2\text{-}\alpha\text{g}^*\text{r}$ closed. This is a contradiction to our assumption. Therefore X is pairwise $\alpha\text{G}^*\text{RO}$ -connected.

Theorem 5.5: If A is a pairwise $\alpha\text{G}^*\text{RO}$ -connected subset of a bitopological space (X, τ_1, τ_2) which has the pairwise $\alpha\text{G}^*\text{RO}$ -separation $X = C \cup D$, then $A \subseteq C$ or $A \subseteq D$.

Proof: Suppose that (X, τ_1, τ_2) has the pairwise $\alpha\text{G}^*\text{RO}$ -separation $X = C \cup D$. Then $X = C \cup D$ where C and D are two nonempty disjoint sets such that $[C \cap \tau_1\text{-}\alpha\text{g}^*\text{rcl}(D)] \cup [\tau_2\text{-}\alpha\text{g}^*\text{rcl}(C) \cap D] = \emptyset$. Since $C \cap D = \emptyset$, we have $C = D^c$ and $D = C^c$. Now $[(C \cap A) \cap \tau_1\text{-}\alpha\text{g}^*\text{rcl}(D \cap A)] \cup [\tau_2\text{-}\alpha\text{g}^*\text{rcl}(C \cap A) \cap (D \cap A)] \subseteq [C \cap \tau_1\text{-}\alpha\text{g}^*\text{rcl}(D)] \cup [\tau_2\text{-}\alpha\text{g}^*\text{rcl}(C) \cap D] = \emptyset$. Hence $A = (C \cap A) \cup (D \cap A)$ is a pairwise $\alpha\text{G}^*\text{RO}$ -separation of A . Since A is pairwise $\alpha\text{G}^*\text{RO}$ -connected, we have either $(C \cap A) = \emptyset$ or $(D \cap A) = \emptyset$. Consequently, $A \subseteq C^c$ or $A \subseteq D^c$. Therefore, $A \subseteq C$ or $A \subseteq D$.

Theorem 5.6: If A is pairwise $\alpha\text{G}^*\text{RO}$ -connected and $A \subseteq B \subseteq \tau_1\text{-}\alpha\text{g}^*\text{rcl}(A) \cap \tau_2\text{-}\alpha\text{g}^*\text{rcl}(A)$ then B is pairwise $\alpha\text{G}^*\text{RO}$ -connected.

Proof: Suppose that B is not pairwise $\alpha\text{G}^*\text{RO}$ -connected. Then $B = C \cup D$ where C and D are two nonempty disjoint sets such that $[C \cap \tau_1\text{-}\alpha\text{g}^*\text{rcl}(D)] \cup [\tau_2\text{-}\alpha\text{g}^*\text{rcl}(C) \cap D] = \emptyset$. Since A is pairwise $\alpha\text{G}^*\text{RO}$ -connected by Theorem 5.5, we have $A \subseteq C$ or $A \subseteq D$. Suppose $A \subseteq C$. Then $D = D \cap B \subseteq D \cap \tau_2\text{-}\alpha\text{g}^*\text{rcl}(A) \subseteq D \cap \tau_2\text{-}\alpha\text{g}^*\text{rcl}(C) = \emptyset$. Consequently, $D = \emptyset$. Similarly, we can prove $C = \emptyset$ if $A \subseteq D$. This is a contradiction to the fact that C and D are nonempty. Therefore, B is pairwise $\alpha\text{G}^*\text{RO}$ -connected.

Theorem 5.7: The union of any family of pairwise $\alpha\text{G}^*\text{RO}$ -connected sets having a nonempty intersection is pairwise $\alpha\text{G}^*\text{RO}$ -connected.

Proof: Let I be an index set and $i \in I$. Let $A = \cup A_i$, where each A_i is pairwise $\alpha\text{G}^*\text{RO}$ -connected with $\cap A_i \neq \emptyset$. Suppose that A is not pairwise $\alpha\text{G}^*\text{RO}$ -connected. Then $A = C \cup D$, where C and D are two nonempty disjoint sets such that $[C \cap \tau_1\text{-}\alpha\text{g}^*\text{rcl}(D)] \cup [\tau_2\text{-}\alpha\text{g}^*\text{rcl}(C) \cap D] = \emptyset$. Since A_i is pairwise $\alpha\text{G}^*\text{RO}$ -connected and $A_i \subset A$, by Theorem 5.5, we have $A_i \subseteq C$ or $A_i \subseteq D$. Since $\cap A_i \neq \emptyset$, there exists an element $x \in \cap A_i$. Therefore, $x \in A_i$ for all i . Suppose $A_i \subseteq C$ for some i . Then $x \in C$ since $C \cap D = \emptyset$, $x \notin D$. Hence, we get $A_j \subseteq C$, for every j . Therefore, $\cup A_i \subseteq C$. Therefore, $A \subseteq C$. Hence $D = \emptyset$. This is a contradiction. Therefore, A is pairwise $\alpha\text{G}^*\text{RO}$ -connected.

6 Pairwise $\alpha\text{G}^*\text{RO}$ -compact spaces

Definition 6.1: A nonempty collection $\mathcal{A} = \{A_i, i \in I, \text{ an index set}\}$ is called a *pairwise regular open cover* of a bitopological space (X, τ_1, τ_2) if $X = \cup A_i$ and $\mathcal{A} \subseteq \tau_1\text{-RO}(X, \tau_1, \tau_2) \cup \tau_2\text{-RO}(X, \tau_1, \tau_2)$ and \mathcal{A} contains at least one member of $\tau_1\text{-RO}(X, \tau_1, \tau_2)$ and one member of $\tau_2\text{-RO}(X, \tau_1, \tau_2)$.

Definition 6.2: A bitopological space (X, τ_1, τ_2) is *pairwise regular compact* if every pairwise regular open cover of X has a finite subcover.

Definition 6.3: A nonempty collection $\mathcal{A} = \{A_i, i \in I, \text{ an index set}\}$ is called a $\tau_i\text{-}\alpha\text{g}^*\text{r}$ open cover of a bitopological space (X, τ_1, τ_2) if $X = \cup A_i$ and $\mathcal{A} \subseteq \tau_i\text{-}\alpha\text{G}^*\text{RO}(X, \tau_1, \tau_2)$, $i = 1, 2$.

Definition 6.4: A bitopological space (X, τ_1, τ_2) is $\tau_i\text{-}\alpha\text{G}^*\text{RO}$ -compact if every $\tau_i\text{-}\alpha\text{g}^*\text{r}$ open cover of X has a finite subcover.

Definition 6.5: A nonempty collection $\mathcal{A} = \{A_i, i \in I, \text{ an index set}\}$ is called a *pairwise $\alpha\text{g}^*\text{r}$ -open cover* of a bitopological space (X, τ_1, τ_2) if $X = \cup A_i$ and $\mathcal{A} \subseteq \tau_1\text{-}\alpha\text{G}^*\text{RO}(X, \tau_1, \tau_2) \cup \tau_2\text{-}\alpha\text{G}^*\text{RO}(X, \tau_1, \tau_2)$ and \mathcal{A} contains at least one member of $\tau_1\text{-}\alpha\text{G}^*\text{RO}(X, \tau_1, \tau_2)$ and one member of $\tau_2\text{-}\alpha\text{G}^*\text{RO}(X, \tau_1, \tau_2)$.

Definition 6.6: A bitopological space (X, τ_1, τ_2) is *pairwise $\alpha\text{G}^*\text{RO}$ -compact* if every pairwise $\alpha\text{g}^*\text{r}$ -open cover of X has a finite subcover.

Theorem 6.7: Every pairwise $\alpha\text{G}^*\text{RO}$ -compact space is pairwise regular compact.

Proof: Let (X, τ_1, τ_2) be pairwise $\alpha\text{G}^*\text{RO}$ -compact. Let $\mathcal{A} = \{A_i, i \in I\}$ be a pairwise regular open cover of X . Then $X = \cup A_i$ and $\mathcal{A} \subseteq \tau_1\text{-RO}(X, \tau_1, \tau_2) \cup \tau_2\text{-RO}(X, \tau_1, \tau_2)$ and \mathcal{A} contains at least one member of $\tau_1\text{-RO}(X, \tau_1, \tau_2)$ and one member of $\tau_2\text{-RO}(X, \tau_1, \tau_2)$. Since every τ_i -regular open set is $\tau_i\text{-}\alpha\text{g}^*\text{r}$ open, we have $X = \cup A_i$ and $\mathcal{A} \subseteq \tau_1\text{-}\alpha\text{G}^*\text{RO}(X, \tau_1, \tau_2) \cup \tau_2\text{-}\alpha\text{G}^*\text{RO}(X, \tau_1, \tau_2)$ and \mathcal{A} contains at least one member of $\tau_1\text{-}\alpha\text{G}^*\text{RO}(X, \tau_1, \tau_2)$ and one member of $\tau_2\text{-}\alpha\text{G}^*\text{RO}(X, \tau_1, \tau_2)$.

$\alpha G^*RO(X, \tau_1, \tau_2)$. Therefore, \mathcal{A} is a pairwise αg^*r -open cover of X . Since X is pairwise αG^*RO -compact, we have \mathcal{A} has a finite subcover. Therefore, X is pairwise regular compact.

Theorem 6.8: If Y is a τ_1 - αg^*r closed subset of a pairwise αG^*RO -compact space (X, τ_1, τ_2) , then Y is τ_2 - αG^*RO compact.

Proof: Let (X, τ_1, τ_2) be a pairwise αG^*RO -compact space. Let $\mathcal{A} = \{A_i, i \in I\}$ be a τ_2 - αg^*r open cover of Y . Since Y is τ_1 - αg^*r closed, Y^c is τ_1 - αg^*r open. Then, $Y^c \cup \mathcal{A} = Y^c \cup \{A_i, i \in I\}$ is a pairwise αg^*r -open cover of X . Since X is pairwise αG^*RO -compact, $X = Y^c \cup A_1 \cup A_2 \cup \dots \cup A_n$. Hence $Y = A_1 \cup A_2 \cup \dots \cup A_n$. Therefore Y is τ_2 - αG^*RO compact.

Theorem 6.9: If Y is a τ_1 -regular closed subset of a pairwise αG^*RO -compact space (X, τ_1, τ_2) , then Y is τ_2 - αG^*RO compact.

Proof: Let Y be a τ_1 -regular closed subset of (X, τ_1, τ_2) . Since every τ_1 -regular closed set is τ_1 - αg^*r closed, we get Y is τ_1 - αg^*r closed. Hence by Theorem 6.8, Y is τ_2 - αG^*RO compact.

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