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THE FORCING MONOPHONIC NUMBER OF A GRAPH

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ABSTRACT

For a connected graph $G = (V,E)$, let a set S be a minimum monophonic set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum monophonic set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing monophonic *number of S, denoted by $f_m(S)$, is the cardinality of a minimum forcing subset of S. The forcing monophonic number of G, denoted by $f_m(G)$, is $f_m(G)=\min\{f_m(S)\}$, where the minimum is taken over all minimum monophonic sets in G. Some general properties satisfied by this concept are studied. The forcing monophonic number of certain classes of graphs are determined. It is shown that for every integers a and b with $0 \le a<b$ and $b>a+1$, there exists a connected graph G such that $f_m(G)=a$ and $m(G) = b$.*

Keywords: *monophonic path, monophonic number, forcing monophonic number.*

AMS Subject Classification: *05C12.*

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1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Harary [1, 6]. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of shortest $u – v$ path in G. An $u – v$ path of length $d(u, v)$ is called an u-v geodetic. A vertex x is said to be lie a u- v geodetic P if x is a vertex of P including the vertices u and v. A geodetic set of G is a set $S \subseteq V$ such that every vertex of G is contained in geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a *minimum geodetic* set or simply a *g-set* of G. The geodetic number of a graph was introduced in [1] and further studied in [2, 7]. It was shown in [7] that determining the geodetic number of a graph is NP – hard problem. A vertex v of G is said to be a geodetic vertex of G if v belongs to every minimum geodetic set of G. A subset $T \subseteq W$ is called a *forcing subset* for W if W is the unique minimum geodetic set containing T. A forcing subset for W of minimum cardinality is a *minimum forcing subset of* W. The *forcing geodetic number* of W, denoted by $f_g(W)$, is the cardinality of a minimum forcing subset of W. The *forcing geodetic number* of G, denoted by $f_g(G)$,is $f_g(G)=\min\{f_g(W)\}$, where the minimum is taken over all minimum geodetic sets W in G. The forcing geodetic number of a graph was introduced in [3].

A *chord* of a path $u_o, u_1, \ldots, u_n$ is an edge u_iu_j with $j \ge i + 2$. $(0 \le i, j \le n)$. An $u – v$ path P is called *monophonic path* if it is a chordless path. A *monophonic set* of G is a set $S \subseteq V$ such that every vertex of G is contained in a *monophonic* path joining some pair of vertices in M. The *monophonic* number $m(G)$ of G is the minimum order of its *monophonic* sets and any monophonic set of order $m(G)$ is a minimum monophonic set or simply a *m-set* of G. The *monophonic* number of a graph is introduced in [5] and further studied in [2, 8, 9]. A vertex v is said to be monophonic vertex of G if v belongs to every minimum monophonic set of G. A vertex v is an extreme vertex of a graph G if the sub graph induced by its neighbours is complete. Throughout the following G denotes a connected graph with at least two vertices.
The following theorems are used in the sequel.

Theorem 1.1: **[**8**]** If v is the extreme vertex of a connected graph G, then G belongs to every *monophonic* set of G.

Theorem 1.2: **[**8**]** For a connected graph G, $m(G) = p$ if and only if $G = K_p$.

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2. THE FORCING *MONOPHONIC* NUMBER OF A GRAPH

Even though every connected graph contains a minimum monophonic set and some connected graphs may contain several minimum monophonic sets. For each minimum monophonic set S in a connected graph G, there is always some subset T of S that uniquely determines S as the minimum monophonic set containing T. Such “forcing subsets” will be considered in this section.

Definition 2.1: Let G be a connected graph and S be a minimum monophonic set of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique minimum monophonic set containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset of S*. The *forcing monophonic number* of S, denoted by $f_m(S)$, is the cardinality of a minimum forcing subset of S. The *forcing monophonic number* of G, denoted by $f_m(G)$, is $f_m(G) = \min\{f_m(S)\}$, where the minimum is taken over all minimum monophonic sets S in G.

Example 2.2: For the graph G given in Figure2.1, $S = \{v_1, v_5, v_{10}\}$ is the unique G-set of G so that $f_g(G) = 0$. Also $S_1 = \{v_1, v_5, v_{10}\}$, $S_2= \{v_1, v_6, v_{10}\}$ and $S_3= \{v_1, v_4, v_{10}\}$ are the only three m-sets of G such that $f_m(S_1)= f_m(S_2)= f_m(S_3)=1$ and so $f_m(G) = 1$.

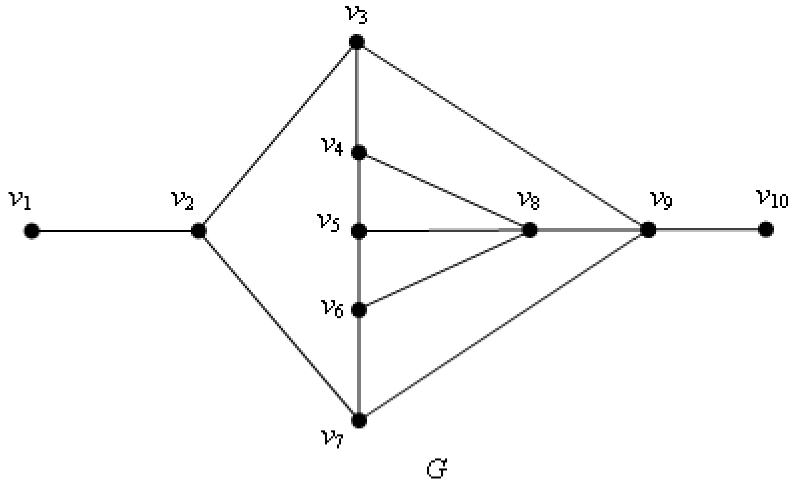

Figure: 2.1

The next theorem follows immediately from the definition of the *monophonic* number and the forcing *monophonic* number of a connected graph G.

Theorem 2.3: For every connected graph G, $0 \leq f_m(G) \leq m(G)$.

Theorem 2.4: Let G be a connected graph. Then
(a) $f_m(G) = 0$ if and only if G has a unique minimum monophonic set.
(b) $f_m(G) = 1$ if and only if G has at least two minimum monophonic sets, one of which is a unique minimum monophonic set containing one of its elements, and
(c) $f_m(G) = m(G)$ if and only if no minimum monophonic set of G is the unique minimum monophonic set containing any of its proper subsets.

Proof: (a) Let $f_m(G) = 0$. Then, by definition, $f_m(S) = 0$ for some minimum monophonic set S of G so that the empty set ϕ is the minimum forcing subset for S. Since the empty set ϕ is a subset of every set, it follows that S is the unique minimum monophonic set of G. The converse is clear.

(b) Let $f_m(G) = 1$. Then by Theorem 2.4(a), G has at least two minimum monophonic sets. Also, since $f_m(G) = 1$, there is a singleton subset T of a minimum monophonic set S of G such that T is not a subset of any other minimum monophonic set of G. Thus S is the unique minimum monophonic set containing one of its elements. The converse is clear.

(c) Let $f_m(G) = m(G)$. Then $f_m(S) = m(G)$ for every minimum monophonic set S in G. Also, by Theorem 2.3, $m(G) \geq 2$ and hence $f_m(G) \geq 2$. Then by Theorem 2.4(a), G has at least two minimum monophonic sets and so the empty set ϕ is not a forcing subset for any minimum monophonic set of G. Since $f_m(S) = m(G)$, no proper subset of S is a forcing subset of S. Thus no minimum monophonic set of G is the unique minimum monophonic set containing any of its proper subsets. Conversely, the data implies that G contains more than one minimum monophonic set and no subset of any minimum monophonic set S other than S is a forcing subset for S. Hence it follows that $f_m(G) = m(G)$.

Definition 2.5: A vertex v of a graph G is said to be a *monophonic vertex* of G if v belongs to every minimum monophonic set of G.

Theorem 2.6: Let G be a connected graph and S be the set of all *monophonic* vertices of G. Then $f_m(G) \leq m(G) - |S|$.

Proof: Let W be any minimum monophonic set of G. Then $m(G) = |W|$, $S \subseteq W$ and W is the unique minimum monophonic set containing $W - S$.

Thus $f_m(G) \leq |W - S| = |W| - |S| = m(G) - |S|$.

Corollary 2.7: If G is a connected graph with k extreme vertices, then

$f_m(G) \leq m(G) - k$.

Proof: This follows from Theorem 1.1 and Theorem 2.6.

Theorem 2.8: For any complete graph $G = K_p(p \geq 2)$ or any non-trivial tree $G = T$, $f_m(G) = 0$.

Proof: For $G = K_p$, it follows from Theorem 1.2 that the set of all vertices of G is the unique monophonic set. Hence it follows from Theorem 2.4(a) that $f_m(G) = 0$. For any non-trivial tree G, the *monophonic* number $m(G)$ equals the number of end vertices in G. In fact, the set of all end vertices of G is the unique minimum monophonic set of G and so $f_m(G) = 0$ by Theorem 2.4(a).

Theorem 2.9: For any cycle $G = C_p(p \geq 4)$, $S = \{u,v\}$ is a minimum monophonic set of G if and only if u and v are independent.

Proof: Let $S = \{u, v\}$, be a minimum monophonic set of G. Suppose that u and v are adjacent. Then uv is a chord for the path u-v and so $\{u, v\}$ is not a monophonic set of G, which is a contradiction. Conversely, let $S = \{u, v\}$, where u and v are independent. It is clear that S is a monophonic set of G. Since $|S| = 2$, S is a minimum monophonic set of G.

Theorem 2.10: For any cycle $G = C_p(p \geq 5)$, $f_m(G) = 2$

Proof: By Theorem 2.8, $m(G) = 2$ and by Theorem 2.3, $0 \leq f_m(G) \leq 2$. Suppose $0 \leq f_m(G) \leq 1$. Since $m(G) = 2$ and the m-set of G is not unique by Theorem 2.4 (b), $fm(G) = 1$. Let $S = \{u, v\}$, be a m-set of G. Let us assume that $fm(S) = 1$. By Theorem 2.4 (b), S is the only *m- set* containing u or v. Let us assume that S is the only m-set containing u. By Theorem 2.9, u is adjacent to more than two vertices of G. Which is a contradiction to G is a cycle.
Therefore $f_m(G) = 2$.

In view of Theorem 2.3, we have the following realization result.

Theorem 2.11: For every integers a and b with $0 \leq a < b$ and $b > a+1$, there exists a connected graph G such that $f_m(G)=a$ and $m(G) = b$.

Proof: If $a = 0$, let $G = K_b$. Then by Theorem 2.8, $f_m(G) = 0$ and by Theorem1. 2, $m(G) = b$. Thus, we assume that $0 < a < b$ and $b > a+1$. Let $F_i : s_i, t_i, v_i, u_i, s_i (1 \leq i \leq a)$ be a copy of cycle C_4. Let H be the graph obtained from F_i's by identifying the vertices t_{i-1} of F_{i-1} and s_i of $F_i(2 \leq i \leq a)$. The graph G given in Figure 2.2 is obtained from H by adding new vertices $x, z_1, z_2, \ldots, z_{b-a-1}$ and joining the edges $xs_1, t_az_1, t_az_2, \ldots, t_az_{b-a-1}$. Let $Z = \{x, z_1, z_2, \ldots, z_{b-a-1}\}$ be the set of end-vertices of G. First we show that $m(G) = b$. Let M be any monophonic set of G. Then by Theorem 1.1, $Z \subseteq S$. It is clear that Z is not a monophonic set of G. Let $H_i = \{v_i, u_i\} (1 \leq i \leq a)$. We observe that every m-set of G must contain at least one vertex from each H_i $(1 \leq i \leq a)$. Thus $m(G) \geq b - a + a = b$. On the other hand since the set $W = Z \cup \{v_1, v_2, v_3, \ldots, v_a\}$ is a monophonic set of G, it follows that $m(G) \leq |W| = b$. Hence $m(G) = b$. Next we show that $f_m(G) = a$. By Theorem 1.1, every monophonic set of G contains Z and so it follows from Theorem 2.6 that $f_m(G) \leq m(G) - |Z| = a$. Now, since $m(G) = b$ and every m-set of G contains Z, it is easily seen that every m-set S is of the form $Z \cup \{c_1, c_2, c_3, \ldots c_a\}$, where $c_i \in H_i(1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then it is clear that there exists some j such that $T \cap H_j = \Phi$, which shows that $f_m(G) = a$

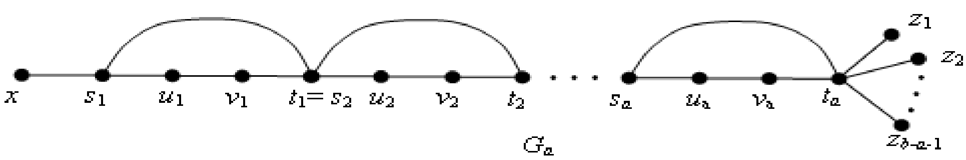

Figure: 2.2

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