International Journal of Mathematical Archive-3(3), 2012, Page: 902-908
Available online through www.ijma.info ISSN 2229 – 5046

RELATED FIXED POINT THEOREM ON N METRIC SPACES

Vishal Gupta[1]*, Ashima Kanwar[2] and R. K. Saini[3]

[1]Department of Mathematics, Maharishi Markandeshwar University, Mullana, Ambala, Haryana (India)
E-mail: [1]vishal.gmn@gmail.com, vkgupta09@rediffmail.com

[2]Department of Mathematics, Maharishi Markandeshwar University, Mullana, Ambala, Hayana (India)
E-mail: [2]kanwar.ashima87@yahoo.com

[3]Department of Mathematics, Bundelkhand University, Jhansi (U.P), (India)
E-mail: rksaini03@yahoo.com

(Received on: 22-02-12; Accepted on: 15-03-12)

ABSTRACT

The aim of this paper to discuss about related fixed point theorem on n metric spaces. Generally, Fixed point theorems on two,three, four and five complete metric spaces have been proved by many authors. In this paper, we prove a new fixed point theorem on metric space using the concept of continous mapping and cauchy sequence on complete metric spaces. Our theorem generalize and extends many result.

1. INTRODUCTION

In this paper, we are dealing with n metric space and we prove a fixed point theorem on n metric space..

In 2009, Faycal Merghadi, Abdelkrim Aliouche, Brain Fisher proved. a fixed point theorem in n complete metric spaces using the concept of implicit relation.

In 2010, Brat Ojha also proved common fixed point theorem for n metric spaces.

Now, in this paper we proved a new fixed point theorem on n metric spaces using continous mapping.

2. PRELIMINARIES

Definition 2.1: Let (X,d) be a metric space. A sequence in X_n is a function $x:\mathbb{N}\mapsto X$.
The image $x(n)$ of $n\in\mathbb{N}$ is usually dentoed by x_n, called the nth term of the sequence $\{x_n\}$.

Definition 2.2: Let (X,d_1) and (Y,d_2) be two metric spaces and let $f:X\mapsto Y$ be a mapping. Then f is said to be continuous at $x\in X$ if and only if the following criterion is satisfied: $\forall\ \varepsilon>0\ \exists$ a positive number $\delta(\varepsilon,x)$ such that $d_2\left(f(x),f(y)\right)<\varepsilon$ for all points $y\in X$ satisfying $d_1(x,y)<\delta$.
If f is continuous at every point of X, then f is called continuous function.

Definition 2.3: A sequence $\{x_n\}$ in a metric space (X,d) is said to be Cauchy sequence if and only iff $\forall\ \varepsilon>0\ \exists$ a postive integer $n_0(\varepsilon)$ such that $d\left(x_m,x_n\right)<\varepsilon,\ \forall\ n,m\geqslant n_0$

Definition 2.4: A metric space (X,d) is said to be complete if and only if every Cauchy sequence in X converges to a point of X.

Corresponding author: Vishal Gupta[1]*, *E-mail: vishal.gmn@gmail.com

3. MAIN RESULT

Theorem 3.1: Let (X_i, d_i) be n complete metric space and T_i be n mapping and $T_1, T_2, T_3, ..., T_{n-1}$ are continuous mapping such that $T_i : X_i \mapsto X_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $T_n : X_n \mapsto X_1$ and satisfying the following inequalities:

$$d_1\left(T_nT_{n-1}\ldots T_2T_1\left(x_2^1\right), T_nT_{n-1}\ldots T_2T_1\left(x^1\right)\right) \leqslant \frac{cF_1\left(x^1, x_2^1\right)}{G_1\left(x^1, x_2^1\right)} \tag{3.1}$$

$$d_2\left(T_1T_n\ldots T_3T_2\left(x_2^2\right), T_1T_n\ldots T_3T_2\left(x^2\right)\right) \leqslant \frac{cF_2\left(x^2, x_2^2\right)}{G_2\left(x^2, x_2^2\right)} \tag{3.2}$$

$$d_3\left(T_2T_1T_n\ldots T_4T_3\left(x_2^3\right), T_2T_1T_n\ldots T_5T_4T_3\left(x^3\right)\right) \leqslant \frac{cF_3\left(x^3, x_2^3\right)}{G_3\left(x^3, x_2^3\right)} \tag{3.3}$$

$$d_n\left(T_{n-1}T_{n-2}\ldots T_1T_n\left(x_2^n\right), T_{n-1}T_{n-2}\ldots T_1T_n\left(x^n\right)\right) \leqslant \frac{cF_n\left(x^n, x_2^n\right)}{G_n\left(x^n, x_2^n\right)} \tag{3.4}$$

$\forall\ x^1, x_2^1 \in X_1, x^2, x_2^2 \in X_2, x^3, x_2^3 \in X_3, \ldots, x^n, x_2^n \in X_n$

for which $G_1\left(x^1, x_2^1\right) \neq 0, G_2\left(x^2, x_2^2\right) \neq 0, G_3\left(x^3, x_2^3\right) \neq 0, \ldots, G_n\left(x^n, x_2^n\right) \neq 0$ where $0 \leqslant c < 1$.

$$\begin{aligned} F_1\left(x^1, x_2^1\right) = \max\Big\{ & d_1\left(x^1, T_nT_{n-1}\ldots T_2T_1x^1\right) d_n\left(T_{n-1}T_{n-2}\ldots T_2T_1x_2^1, T_{n-1}T_{n-2}\ldots T_2T_1x^1\right), \\ & d_1\left(x^1, T_nT_{n-1}\ldots T_2T_1x^1\right) d_{n-1}\left(T_{n-2}T_{n-3}\ldots T_2T_1x_2^1, T_{n-2}T_{n-3}\ldots T_2T_1x^1\right), \\ & \ldots, d_1\left(x^1, T_nT_{n-1}\ldots T_2T_1x^1\right) d_3\left(T_2T_1x_1^2, T_2T_1x^1\right), d_1\left(x^1, T_nT_{n-1}\ldots T_2T_1x^1\right) \\ & d_2\left(T_1x_2^1, T_1T_n\ldots T_3T_2T_1x_2^1\right), d_1\left(x^1, T_nT_{n-1}\ldots T_2T_1x_2^1\right) d_2\left(T_1x_2^1, T_1x^1\right)\Big\} \end{aligned}$$

$$\begin{aligned} F_2\left(x^2, x_2^2\right) = \max\Big\{ & d_2\left(x^2, T_1T_n\ldots T_3T_2x^2\right) d_1\left(T_nT_{n-1}\ldots T_3T_2x_2^2, T_nT_{n-1}\ldots T_3T_2x^2\right), d_2\left(x^2, T_1T_n\ldots T_3T_2x^2\right) \\ & d_n\left(T_{n-1}T_{n-2}\ldots T_4T_3x^3, T_{n-1}T_{n-2}\ldots T_3T_2x^2\right), d_2\left(x^2, T_1T_n\ldots T_3T_2x^2\right) \\ & d_{n-1}\left(T_{n-2}T_{n-3}\ldots T_3T_2x_2^2, T_{n-2}T_{n-3}\ldots T_3T_2x^2\right), \ldots, d_2\left(x^2, T_1T_n\ldots T_3T_2x^2\right) \\ & d_3\left(T_2x_2^2, T_2T_1T_n\ldots T_3T_2x_2^2\right), d_2\left(x^2, T_1T_n\ldots T_3T_2x_2^2\right) d_3\left(T_2x_2^2, T_2x^2\right)\Big\} \end{aligned}$$

$$\begin{aligned} F_3\left(x^3, x_2^3\right) = \max\Big\{ & d_3\left(x^3, T_2T_1T_n\ldots T_4T_3x^3\right) d_2\left(T_1T_n\ldots T_4T_3x_2^3, T_1T_n\ldots T_4T_3x^3\right), d_3\left(x^3, T_2T_1T_n\ldots T_4T_3x^3\right) \\ & d_1\left(T_nT_{n-1}\ldots T_4T_3x_2^3, T_nT_{n-1}\ldots T_4T_3x^3\right), d_3\left(x^3, T_2T_1T_n\ldots T_4T_3x^3\right) \\ & d_n\left(T_{n-1}T_{n-2}\ldots T_4T_3x_2^3, T_{n-1}T_{n-2}\ldots T_4T_3x^3\right), d_3\left(x^3, T_2T_1T_n\ldots T_4T_3x^3\right) \\ & d_{n-1}\left(T_{n-2}T_{n-3}\ldots T_4T_3x_2^3, T_{n-2}T_{n-3}\ldots T_4T_3x^3\right), \ldots, d_3\left(x^3, T_2T_1T_n\ldots T_4T_3x^3\right) \\ & d_4\left(T_3x_2^3, T_3T_2T_1T_n\ldots T_4T_3x_2^3\right), d_3\left(x^3, T_2T_1T_n\ldots T_4T_3x_2^3\right) d_4\left(T_3x_2^3, T_3x^3\right)\Big\} \end{aligned}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\begin{aligned} F_n\left(x^n, x_2^n\right) = \max\Big\{ & d_n\left(x^n, T_{n-1}T_{n-2}\ldots T_1T_nx^n\right) d_{n-1}\left(T_{n-2}T_{n-3}\ldots T_1T_nx_2^n, T_{n-2}T_{n-3}\ldots T_1T_nx^n\right), \\ & d_n\left(x^n, T_{n-1}T_{n-2}\ldots T_1T_nx^n\right) d_{n-2}\left(T_{n-3}T_{n-4}\ldots T_1T_nx_2^n, T_{n-3}T_{n-4}\ldots T_1T_nx^n\right), \ldots, \\ & d_n\left(x^n, T_{n-1}T_{n-2}\ldots T_1T_nx^n\right) d_1\left(T_nx_2^n, T_nT_{n-1}\ldots T_2T_1x^1\right), d_n\left(x^n, T_{n-1}T_{n-2}\ldots T_1T_nx_2^n\right) d_1\left(T_nx_2^n, T_nx^n\right)\Big\} \end{aligned}$$

And

$$G_1\left(x^1,x_2^1\right)=\max\left\{d_1\left(x^1,T_nT_{n-1}\dots T_2T_1x_2^1\right),d_1\left(x^1,T_nT_{n-1}\dots T_2T_1x^1\right),d_2\left(T_1x^1,T_1T_n\dots T_2T_1x_2^1\right)\right\}$$

$$G_2\left(x^2,x_2^2\right)=\max\left\{d_2\left(x^2,T_1T_n\dots T_3T_2x_2^2\right),d_2\left(x^2,T_1T_n\dots T_3T_2x^2\right),d_3\left(T_2x^2,T_2T_1T_n\dots T_3T_2x_2^2\right)\right\}$$

$$\vdots\quad\vdots\quad\vdots$$

$$G_n\left(x^n,x^1\right)=\max\left\{d_n\left(x^n,T_{n-1}T_{n-2}\dots T_1T_nx_1^n\right),d_n\left(x^n,T_{n-1}T_{n-2}\dots T_1T_nx^n\right),d_1\left(T_nx^n,T_nT_{n-1}\dots T_1T_nx_1^n\right)\right\}$$

Then $T_nT_{n-1}\dots T_2T_1$ has a unique fixed point $\alpha_1\in X_1$, $T_1T_n\dots T_3T_2$ has a unique fixed point $\alpha_2\in X_2$, and so on, $T_{n-1}T_{n-2}\dots T_1T_n$ has a unique fixed point $\alpha_n\in X_n$.

Further, $T_1\left(\alpha_1\right)=\alpha_2,T_2\left(\alpha_2\right)=\alpha_3,T_3\left(\alpha_3\right)=\alpha_4,\dots,\ T_{n-1}\left(\alpha_{n-1}\right)=\alpha_n,T\left(\alpha_n\right)=\alpha_1$

Proof: Let $x_0'\in X$, be any arbitrary point. We define sequences $\{x_m^1\}$, $\{x_m^2\}$, $\{x_m^3\},\dots,\{x_m^n\}$ with $X_1,X_2,X_3,\dots,X_n$ respectively as follows:

$$x_m^1=\left(T_nT_{n-1}\dots T_2T_1\right)^m x_0^1,x_m^2=T_1x_{m-1}^1,\ \ x_m^3=T_2\left(x_m^2\right),x_m^4=T_3\left(x_m^3\right)\forall\ m\in\mathbb{N}$$

We assume that $x_m^1\neq x_{m+1}^1,x_m^2\neq x_{m+1}^2x_m^3\neq x_{m+1}^3,\ \dots x_m^n\neq x_{m+1}^n\forall\ m\in\mathbb{N}$

Taking $x^2=x_m^2$ and $x_2^2=x_{m-1}^2$ in inequality (3.2), we get,

$$d_2\left(x_m^2,x_{m+1}^2\right)=d_2\left(T_1T_n\dots T_3T_2x_{m-1}^2,T_1T_nT_{n-1}\dots T_3T_2\left(x_m^2\right)\right)$$

$$\leqslant\frac{cF_2\left(x_m^2,x_{m-1}^2\right)}{G_2\left(x_m^2,x_{m-1}^2\right)}$$

$$=\frac{\begin{array}{l}c\max\left\{d_2\left(x_m^2,T_1T_n\dots T_3T_2x_m^2\right)d_1\left(T_nT_{n-1}\dots T_3T_2x_{m-1}^2,T_nT_{n-1}\dots T_2x_m^2\right),d_2\left(x_m^2,T_1T_n\dots T_3T_2x_m^2\right)\right.\\ d_n\left(T_{n-1}T_{n-2}\dots T_3T_2x_{m-1}^2,T_{n-1}\dots T_3T_2x_m^2\right),d_2\left(x_m^2,T_1T_n\dots T_3T_2x_m^2\right)d_{n-1}\left(T_{n-2}T_{n-3}\dots T_3T_2x_{m-1}^2,T_{n-2}\dots T_3T_2x_m^2\right),\dots,\\ \left.d_2\left(x_m^2,T_1T_n\dots T_3T_2x_m^2\right)d_3\left(T_2x_{m-1}^2,T_2T_1T_n\dots T_2x_{m-1}^2\right),d_2\left(x_m^2,T_1T_n\dots T_3T_2x_{m-1}^2\right)d_3\left(T_1x_{m-1}^2,T_2x_m^2\right)\right\}\end{array}}{\max\left\{d_2\left(x_m^2,x_m^2\right),d_2\left(x_m^2,x_{m+1}^2\right),d_3\left(x_m^3,x_m^3\right)\right\}}$$

$$d_2\left(x_m^2,x_{m+1}^2\right)\leqslant c\max\left\{d_1\left(x_{m-1}^1,x_m^1\right),d_n\left(x_{m-1}^n,x_m^n\right),d_{n-1}\left(x_{m-1}^{n-1},x_m^{n-1}\right),\dots,d_3\left(x_{m-1}^3,x_m^3\right)\right\}\tag{3.5}$$

Now taking $x^3=x_m^3$ and $x_2^3=x_{m-1}^3$ in inequality (3.3), we obtain,

$$d_3\left(x_m^3,x_{m+1}^3\right)=d_3\left(T_2T_1T_n\dots T_4T_3x_{m-1}^3,T_2T_1T_n\dots T_4T_3x_m^3\right)\leqslant\frac{cF_3\left(x_m^3,x_{m-1}^3\right)}{G_3\left(x_m^3,x_{m-1}^3\right)}$$

$$= \frac{\begin{array}{l} c\max\left\{d_3\left(x_m^3, T_2T_1T_n\ldots T_3x_m^3\right)d_2\left(T_1T_n\ldots T_4T_3x_{m-1}^3, T_1T_n\ldots T_4T_3x_m^3\right), d_3\left(x_m^3, T_2T_1T_n\ldots T_3x_m^3\right)\right. \\ d_1\left(T_n\ldots T_4T_3x_{m-1}^3, T_n\ldots T_4T_3x_m^3\right), d_3\left(x_m^3, T_2T_1T_n\ldots T_4T_3x_m^3\right)d_{n-1}\left(T_{n-2}\ldots T_4T_3x_{m-1}^3, T_{n-2}\ldots T_4T_3x_m^3\right),\ldots, \\ \left. d_3\left(x_m^3, T_2T_1T_n\ldots T_4T_3x_m^3\right)d_4\left(T_3x_{m-1}^3, T_3T_2T_1T_n\ldots T_4T_3x_{m-1}^3\right), d_3\left(x_m^3, T_2T_1T_3\ldots T_4T_3x_{m-1}^3\right)d_4\left(T_3x_{m-1}^3, T_3x_m^3\right)\right\} \end{array}}{\left(\max\left\{d_3\left(x_m^3, x_m^3\right), d_3\left(x_m^3, x_{m+1}^3\right), d_4\left(x_m^4, x_m^4\right)\right\}\right)}$$

$$= c\max\left\{d_2\left(x_m^2, x_{m+1}^2\right), d_1\left(x_{m-1}^1, x_m^1\right), d_n\left(x_{m-1}^n, x_m^n\right), d_{n-1}\left(x_{m-1}^{n-1}, x_m^{n-1}\right),\ldots, d_4\left(x_{m-1}^4, x_m^4\right)\right\}$$

From inequality (3.5), we get,

$$d_3\left(x_m^3, x_{m+1}^3\right) \leqslant \left\{cd_1\left(x_{m-1}^1, x_m^1\right), cd_n\left(x_{m-1}^n, x_m^n\right), cd_{n-1}\left(x_{m-1}^{n-1}, x_m^{n-1}\right),\ldots, cd_3\left(x_{m-1}^3, x_m^3\right), d_1\left(x_{m-1}^1, x_m^1\right)\right.$$
$$\left., d_n\left(x_{m-1}^n, x_m^n\right), d_{n-1}\left(x_{m-1}^{n-1}, x_m^{n-1}\right),\ldots, d_4\left(x_{m-1}^4, x_m^4\right)\right\}$$

Since $0 \leqslant c < 1$, so, we get,

$$d_3\left(x_m^3, x_{m+1}^3\right) \leqslant c\max\left\{d_1\left(x_{m-1}^n, x_m^1\right), d_n\left(x_{m-1}^n, x_m^n\right), d_{n-1}\left(x_{m-1}^{n-1}, x_m^{n-1}\right),\ldots, d_4\left(x_{m-1}^4, x_m^4\right), d_3\left(x_{m-1}^3, x_m^3\right)\right\} \quad (3.6)$$

Similarly, we can get,

$$d_{n-1}\left(x_m^{n-1}, x_{m+1}^{n-1}\right) \leqslant c\max\left\{d_1\left(x_{m-1}^1, x_m^1\right), d_3\left(x_{m-1}^3, x_m^3\right), d_4\left(x_{m-1}^4, x_m^4\right),\ldots,\right.$$
$$\left. d_{n-1}\left(x_{m-1}^{n-1}, x_m^{n-1}\right), d_n\left(x_{m-1}^n, x_m^n\right)\right\} \quad (3.7)$$

Taking $x_2^n = x_{m-1}^n$ and $x^n = x_m^n$ in inequality (3.4)

$$d_n\left(x_m^n, x_{m+1}^n\right) = d_n\left(T_{n-1}T_{n-2}\ldots T_1T_nx_{m-1}^n, T_{n-1}T_{n-2}\ldots T_1T_nx_m^n\right) \leqslant \frac{cF_n\left(x_m^n, x_{m-1}^n\right)}{G_n\left(x_m^n, x_{m-1}^n\right)}$$

$$= \frac{\begin{array}{l} c\max\left\{d_n\left(x_m^n, T_{n-1}T_{n-2}\ldots T_2T_1T_nx_m^n\right)d_{n-1}\left(T_{n-2}\ldots T_1T_nx_{m-1}^n, T_{n-2}\ldots T_1T_nx_m^n\right),\right. \\ d_n\left(x_m^n, T_{n-1}T_{n-2}\ldots T_1T_nx_m^n\right)d_{n-2}\left(T_{n-3}\ldots T_1T_nx_{m-1}^n, T_{n-3}\ldots T_nx_m^n\right),\ldots, d_n\left(x_m^n, T_{n-1}\ldots T_1T_nx_m^n\right) \\ \left. d_1\left(T_nx_{m-1}^n, T_nT_{n-1}\ldots T_1T_nx_{m-1}^n\right), d_n\left(x_m^n, T_{n-1}\ldots T_1T_nx_{m-1}^n\right)d_1\left(T_nx_{m-1}^n, T_nx_m^n\right)\right\} \end{array}}{\max\left\{d_n\left(x_m^n, T_{n-1}T_{n-2}\ldots T_1T_nx_{m-1}^n\right), d_n\left(x_m^n, T_{n-1}T_{n-2}\ldots T_1T_nx_m^n\right), d_1\left(T_nx_m^n, T_nT_{n-1}\ldots T_2T_1T_nx_{m-1}^n\right)\right\}}$$

$$= c\max\left\{d_{n-1}\left(x_m^{n-1}, x_{m+1}^{n-1}\right), d_{n-2}\left(x_m^{n-2}, x_{m+1}^{n-2}\right),\ldots, d_1\left(x_{m-1}^1, x_m^1\right)\right\}$$

From (3.5), (3.6) and (3.7), we get,

$$d_n\left(x_m^n, x_{m+1}^n\right) \leqslant c\max\left\{d_1\left(x_{m-1}^1, x_m^1\right), d_3\left(x_{m-1}^3, x_m^3\right), d_4\left(x_{m-1}^4, x_m^4\right),\ldots, d_{n-1}\left(x_{m-1}^n, x_m^{n-1}\right), d_n\left(x_{m-1}^n, x_m^n\right)\right\} (3\quad)$$

It now follows by induction on above inequalities, we get,

$$d_1\left(x_m^1, x_{m+1}^1\right) \leqslant c^{m-1}\max\left\{d_1\left(x_1^1, x_2^1\right), d_3\left(x_1^3, x_2^3\right),\ldots, d_{n-1}\left(x_1^{n-1}, x_2^{n-1}\right), d_n\left(x_1^n, x_2^n\right)\right\}$$

$$d_2\left(x_m^2, x_{m+1}^2\right) \leqslant c^{m-1}\max\left\{d_1\left(x_1^1, x_2^1\right), d_3\left(x_1^3, x_2^3\right),\ldots, d_{n-1}\left(x_1^{n-1}, x_2^{n-1}\right), d_n\left(x_1^n, x_1^n\right)\right\}$$

$$\vdots$$

$$d_n\left(x_m^n, x_{m+1}^n\right) \leqslant c^{m-1} \max\left\{d_1\left(x_1^1, x_2^1\right), d_3\left(x_1^3, x_2^3\right), \ldots, d_{n-1}\left(x_1^{n-1}, x_2^{n-1}\right), d_n\left(x_1^n, x_2^n\right)\right\}$$

Since $0 \leqslant c < 1$, the sequence $\left\{x_m^1\right\}, \left\{x_m^2\right\}, \left\{x_m^3\right\}, \ldots, \left\{x_m^{n-1}\right\}, \left\{x_m^n\right\}$ are Cauchy sequence in $X_1, X_2, X_3, \ldots, X_n$. So we have,

$$\lim_{m\to\infty} x_m^i = \alpha_i \text{ when } i = 1, 2, 3, \ldots, n.$$

Since $T_1, T_2, T_3, \ldots, T_{n-1}$ are continuous mapping.

$$T_1\alpha_1 = \alpha_2, T_2\alpha_2 = \alpha_3, ..., T_{n-1}\alpha_{n-1} = \alpha_n$$

Now taking $x^1 = x_{m-1}^1$ and $x_2^1 = \alpha_1$ in inequality (3.1),

$$d_1\left(T_nT_{n-1}\ldots T_2T_1\alpha_1, T_nT_{n-1}\ldots T_2T_1x_m^1\right) \leqslant \frac{cF_1\left(x_{m-1}^1, \alpha_1\right)}{G_1\left(x_{m-1}^1, \alpha_1\right)}$$

$$= \frac{\begin{array}{l} c\max\left\{d_1\left(x_{m-1}^1, T_n\ldots T_2T_1x_{m-1}^1\right) d_n\left(T_{n-1}\ldots T_2T_1\alpha_1, T_{n-1}\ldots T_2T_1x_{m-1}^1\right) d_1\left(x_{m-1}^1, T_n\ldots T_2T_1x_{m-1}^1\right)\right. \\ d_{n-1}\left(T_{n-2}\ldots T_2T_1\alpha_1, T_{n-2}\ldots T_2T_1x_{m-1}^1\right), \ldots, d_1\left(x_{m-1}^1, T_n\ldots T_2T_1x_{m-1}^1\right) d_3\left(T_2T\alpha_1, T_2T_1x_{m-1}^1\right), d_1\left(x_{m-1}^1, T_n\ldots T_2T_1x_{m-1}^1\right) \\ \left. d_2\left(T_1\alpha_1, T_1T_n\ldots T_2T_1\alpha_1\right), d_1\left(x_{m-1}^1, T_1T_nT_{n-1}\ldots T_2T_1\alpha_1\right) d_2\left(T_1\alpha_1, T_1x_{m-1}^1\right)\right\} \end{array}}{\max\left\{d_1\left(x_{m-1}^1, T_nT_{n-1}\ldots T_2T_1\alpha_1\right), d_1\left(x_{m-1}^1, T_nT_{n-1}\ldots T_2T_1, x_{m-1}^1\right), d_2\left(T_1x_{m-1}^1, T_1T_n\ldots T_2T_1\alpha_1\right)\right\}}$$

As $m \to \infty$, we get,

$$d_1\left(T_nT_{n-1}\ldots T_2T_1\alpha_1, \alpha_1\right) \leq \frac{cd_1\left(\alpha_1, T_1T_nT_{n-1}\ldots T_2T_1\alpha_1\right) d_2\left(T_1\alpha_1, \alpha_2\right)}{\max\{d_1\left(\alpha_1, T_nT_{n-1}\ldots T_2T_1\alpha_1\right), d_2\left(\alpha_2, T_1T_n\ldots T_2T_1\alpha_1\right)\}}$$

Since T_1 is continuous mapping.

$$\Rightarrow d_1\left(T_nT_{n-1}\ldots T_2T_1\alpha_1, \alpha_1\right) \leqslant 0$$

Since, which is not possible

$$\Rightarrow T_nT_{n-1}\ldots T_2T_1\alpha_1 = \alpha_1 \tag{3.9}$$

Now, $T_1T_nT_{n-1}\ldots T_3T_2\alpha_2 = T_1T_nT_{n-1}\ldots T_3T_2(T_1\alpha_1) = T_1(T_nT_{n-1}\ldots T_3T_2T_1\alpha_1)$

$$= T_1\alpha_1 = \alpha_2$$

$$\Rightarrow T_1T_nT_{n-1}\ldots T_3T_2\alpha_2 = \alpha_2$$

Since $T_1, T_2, T_3, \ldots, T_{n-1}$ are continuous mapping, So in same above way,so we obtain

$$T_1T_nT_{n-1}\ldots T_2\alpha_2 = \alpha_2$$
$$T_2T_1T_n\ldots T_3\alpha_3 = \alpha_3$$
$$T_3T_2T_1\ldots T_4\alpha_4 = \alpha_4$$
$$\vdots$$
$$T_{n-1}\ldots T_2T_1T_n\alpha_n = \alpha_n$$

and $\alpha_1 \in X_1, \alpha_2 \in X_2, \ldots, \alpha_n \in X_n$ and $T_1\alpha_1 = \alpha_2, T_2\alpha_2 = \alpha_3, \ldots, T_{n-1}\alpha_{n-1} = \alpha_n$.

Uniqueness

Let us assume that α_1' is another fixed point of $T_nT_{n-1}\dots T_2T_1$ different from α_1 such that $\alpha_{1'} \neq \alpha_1$ and

$$T_nT_{n-1}\dots T_2T_1\alpha_1' = \alpha_1',\ T_nT_{n-1}\dots T_2T_1\alpha_1 = \alpha_1.$$

Now taking $x^1 = \alpha_1'$ and $x_2^1 = \alpha_1$ in inequality (3.1), we get,

$$d_1(\alpha,\alpha_1') = d_1(T_nT_{n-1}\dots T_2T_1\alpha_1, T_nT_{n-1}\dots T_1\alpha_1')$$

$$\leqslant \frac{cF_1(\alpha_1',\alpha_1)}{G_1(\alpha_1',\alpha_1)}$$

$$= \frac{\left(\begin{gathered} c\max\{d_1(\alpha_1',T_n\dots T_2T_1\alpha_1')\,d_n(T_{n-1}\dots T_3T_2T_1, T_{n-1}\dots T_2T_1\alpha_1'), d_1(\alpha_1',T_n\dots T_2T_1\alpha_1') \\ d_{n-1}(T_{n-2}\dots T_3T_2T_1\alpha_1, T_{n-2}\dots T_2T_1\alpha_1'),\dots, d_1(\alpha_1',T_n\dots T_2T_1\alpha_1')\,d_3(T_2T_1\alpha_1,T_2T_1\alpha_1'), \\ d_1(\alpha_1',T_n\dots T_2T_1\alpha_1')\,d_2(T_1\alpha_1, T_1T_n\dots T_3T_2T_1\alpha_1), d_1(\alpha_1', T_nT_{n-1}\dots T_3T_2T_1\alpha_1)\,d_2(T_1\alpha_1,T_1\alpha_1')\} \end{gathered}\right)}{\left(\{\max\{d_1(\alpha_1',T_nT_{n-1}\dots T_3T_2T_1\alpha_1),\}\,d_1(\alpha_1',T_nT_{n-1}\dots T_2T_1\alpha_1'), d_2(T_1\alpha_1', T_1T_n\dots T_2T_1\alpha_1)\}\right)}$$

$$= \frac{cd_1(\alpha_1',\alpha_1)\,d_2(T_1\alpha, T_1\alpha_1')}{\max\{d_1(\alpha_1',\alpha_1), d_2(T_1\alpha_1',T_1\alpha_1)\}}$$

$$\Rightarrow d_1(\alpha_1,\alpha_1') \leqslant cd_2(T_1\alpha_1, T_1\alpha_1') \tag{3.10}$$

In the same way, taking $x_2^2 = T_1\alpha_1$ and $x^2 = T_1\alpha_1'$ in inequality (3.2), we get

$$d_2(T_1T_nT_{n-1}\dots T_3T_2T_1\alpha_1, T_1T_nT_{n-1}\dots T_2T_1\alpha_1') \leqslant \frac{cF_2(T_1\alpha_1',T_1\alpha_1)}{G_2(T_1\alpha_1',T_1\alpha_1)}$$

$$= \frac{\left(\begin{gathered} c\max\{d_2(T_1\alpha_1',T_1\alpha_1')\,d_1(T_nT_{n-1}\dots T_2T_1\alpha_1, \\ T_n\dots T_2T_1\alpha_1'),\dots, d_2(T_1\alpha_1',T_1\alpha_1)\,d_3(T_2T_1\alpha_1,T_2T_1\alpha_1')\} \end{gathered}\right)}{\left(\max\{d_2(T_1\alpha_1',T_1\alpha_1), d_2(T_1\alpha_1',T_1\alpha_1'), d_3(T_2T_1\alpha_1',T_2T_1\alpha_1)\}\right)}$$

$$\Rightarrow d_2(T_1\alpha_1,T_1\alpha_1') \leqslant cd_3(T_2T_1\alpha_1, T_2T_1\alpha_1') \tag{3.11}$$

Similarly, by taking $x_2^3 = T_2T_1\alpha_1$ and $x^3 = T_2T_1\alpha_1'$ in inequality (3.3), we get,

$$d_3(T_2T_1\alpha_1, T_2T_1\alpha_1') \leqslant cd_4(T_3T_2T_1\alpha_1, T_3T_2T_1\alpha_1') \tag{3.12}$$

Continue like this, taking $x_2^n = T_{n-1}T_{n-2}\dots T_2T_1\alpha_1$ and $x^n = T_{n-1}\dots T_2T_1\alpha_1'$ in inequality (3.4), we get,

$$d_n(T_{n-1}T_{n-2}\dots T_2T_1\alpha_1, T_{n-1}\dots T_2T_1\alpha_1') \leqslant cd_1(T_nT_{n-1}\dots T_2T_1\alpha_1, T_nT_{n-1}\dots T_2T_1\alpha_1')$$
$$= cd_1(\alpha_1,\alpha_1') \tag{3.13}$$

From inequalities (3.10),(3.11), (3.12)and(3.13), we get,

$$\begin{aligned} d_1(\alpha_1,\alpha_1') &\leqslant cd_2(T_1\alpha_1,T_1\alpha_1') \\ &\leqslant c^2 d_3(T_2T_1\alpha_1,T_2T_1\alpha_1') \\ &\leqslant c^3 d_4(T_3T_2T_1\alpha_1,T_3T_2\alpha_1') \\ &\ \ \vdots \\ &\leqslant c^{n-1} d_n(T_{n-1}\ldots T_2T_1\alpha_1,T_{n-1}\ldots T_2T_1\alpha_1') \\ &\leqslant c^n d_1(\alpha_1,\alpha_1') \end{aligned}$$

$$\Rightarrow d_1(\alpha_1,\alpha_1') \leqslant c^n d_1(\alpha_1,\alpha_1')$$

Since $0 \leqslant c < 1$, so we get,

$$\Rightarrow d_1(\alpha_1,\alpha_1') = 0$$

$$\Leftrightarrow \alpha_1 = \alpha_1'$$

So, α_1 is unique fixed point of $T_nT_{n-1}\ldots T_2T_1$.

In the similar way, the unicity of other fixed point can be proved.

We have to finally proved that $T(\alpha_n) = \alpha_1$, To do this, note that

$$T_n(\alpha_n) = T_n(T_{n-1}\ldots T_2T_1T_n\alpha_n) = T_nT_{n-1}\ldots T_2T_1(T_n\alpha_n)$$

And so $T_n\alpha_n$ is fixed point of $T_nT_{n-1}\ldots T_2T_1$. Since α_1 is the unique fixed point of $T_nT_{n-1}\ldots T_2T_1$. It follows that $T(\alpha_n) = \alpha_1$

So, we obtain $T_nT_{n-1}\ldots T_2T_1$ has a unique fixed point $\alpha_1 \in X_1$, $T_1T_n\ldots T_3T_2$ has a unique fixed point $\alpha_2 \in X_2$, and so on, $T_{n-1}T_{n-2}\ldots T_1T_n$ has a unique fixed point $\alpha_n \in X_n$ Also $T_1(\alpha_1) = \alpha_2, T_2(\alpha_2) = \alpha_3, T_3(\alpha_3) = \alpha_4, \ldots,$ $T_{n-1}(\alpha_{n-1}) = \alpha_n, T(\alpha_n) = \alpha_1$

REFERNCES

[1] F.Merghadi, A.Aliouche and B.Fisher, A related fixed point theorem in n complete metric spaces, *Acta Universitatis Apulensis, no.19(2009), 91-99*

[2] D.B.Ojha , Commom fixed point theorem for n metric spaces, *International Journal of Applied Engineering Research,Dindigul,Vol.1(2010) no. 3, 324-329*

[2] B. Fisher, Fixed points theorem on two metric spaces, *Glasnik Mat., Vol. 16 (1981) no.36, 333-337.*

[2] L. Kikina and K. Kikina, A fixed point theorem on four spaces, *Int. Math. J. Analysis,Vol. 3 (2009)no.32 , 1559-1568.*

[3] N.P. Nung, A fixed point theorem for three metric spaces, *Math. Seminar Notes Kobe University, Vol.1(1983), 77-79.*

[4] R.K. Jain, H.K. Sahu, and B. Fisher, A fixed point theorem on three metric spaces, *Kyungpook Math. J.,Vol. 36 (1996), 151-154.*

[5] S. Jain and B. Fisher, A related fixed point theorems on three complete metric spaces, *Novisad J. Math. ,Vol.27 (1997), 27-35.*

****

