

RELATED FIXED POINT THEOREM ON N METRIC SPACES

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ABSTRACT

The aim of this paper to discuss about related fixed point theorem on n metric spaces. Generally, Fixed point theorems on two, three, four and five complete metric spaces have been proved by many authors. In this paper, we prove a new fixed point theorem on metric space using the concept of continous mapping and cauchy sequence on complete metric spaces. Our theorem generalize and extends many result.

1. INTRODUCTION

In this paper, we are dealing with n metric space and we prove a fixed point theorem on n metric space..

In 2009, Faycal Merghadi, Abdelkrim Aliouche, Brain Fisher proved. a fixed point theorem in n complete metric spaces using the concept of implicit relation.

In 2010, Brat Ojha also proved common fixed point theorem for n metric spaces.

Now, in this paper we proved a new fixed point theorem on n metric spaces using continous mapping.

2. PRELIMINARIES

Definition 2.1: Let (X, d) be a metric space. A sequence in X_n is a function $x : \mathbb{N} \mapsto X$.

The image $x(n)$ of $n \in \mathbb{N}$ is usually denoted by x_n , called the nth term of the sequence $\{x_n\}$.

Definition 2.2: Let (X, d_1) and (Y, d_2) be two metric spaces and let $f : X \mapsto Y$ be a mapping. Then f is said to be continuous at $x \in X$ if and only if the following criterion is satisfied: $\forall \varepsilon > 0 \exists$ a positive number $\delta(\varepsilon, x)$ such that $d_2(f(x), f(y)) < \varepsilon$ for all points $y \in X$ satisfying $d_1(x, y) < \delta$.

If f is continuous at every point of X , then f is called continuous function.

Definition 2.3: A sequence $\{x_n\}$ in a metric space (X, d) is said to be Cauchy sequence if and only iff $\forall \varepsilon > 0 \exists$ a positive integer $n_0(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon, \forall n, m \geq n_0$

Definition 2.4: A metric space (X, d) is said to be complete if and only if every Cauchy sequence in X converges to a point of X .

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3. MAIN RESULT

Theorem 3.1: Let (X_i, d_i) be n complete metric space and T_i be n mapping and $T_1, T_2, T_3, \dots, T_{n-1}$ are continuous mapping such that $T_i : X_i \mapsto X_{i+1}$ for $i = 1, 2, \dots, n-1$ and $T_n : X_n \mapsto X_1$ and satisfying the following inequalities:

$$d_1(T_n T_{n-1} \dots T_2 T_1(x_2^1), T_n T_{n-1} \dots T_2 T_1(x_1^1)) \leq \frac{cF_1(x^1, x_2^1)}{G_1(x^1, x_2^1)} \quad (3.1)$$

$$d_2(T_1 T_n \dots T_3 T_2(x_2^2), T_1 T_n \dots T_3 T_2(x_1^2)) \leq \frac{cF_2(x^2, x_2^2)}{G_2(x^2, x_2^2)} \quad (3.2)$$

$$d_3(T_2 T_1 T_n \dots T_4 T_3(x_2^3), T_2 T_1 T_n \dots T_4 T_3(x_1^3)) \leq \frac{cF_3(x^3, x_2^3)}{G_3(x^3, x_2^3)} \quad (3.3)$$

$$d_n(T_{n-1} T_{n-2} \dots T_1 T_n(x_2^n), T_{n-1} T_{n-2} \dots T_1 T_n(x_1^n)) \leq \frac{cF_n(x^n, x_2^n)}{G_n(x^n, x_2^n)} \quad (3.4)$$

$$\forall x^1, x_2^1 \in X_1, x^2, x_2^2 \in X_2, x^3, x_2^3 \in X_3, \dots, x^n, x_2^n \in X_n$$

for which $G_1(x^1, x_2^1) \neq 0, G_2(x^2, x_2^2) \neq 0, G_3(x^3, x_2^3) \neq 0, \dots, G_n(x^n, x_2^n) \neq 0$ where $0 \leq c < 1$.

$$F_1(x^1, x_2^1) = \max \left\{ d_1(x^1, T_n T_{n-1} \dots T_2 T_1 x_1^1) d_n(T_{n-1} T_{n-2} \dots T_2 T_1 x_2^1, T_{n-1} T_{n-2} \dots T_2 T_1 x_1^1), \right. \\ d_1(x^1, T_n T_{n-1} \dots T_2 T_1 x_1^1) d_{n-1}(T_{n-2} T_{n-3} \dots T_2 T_1 x_2^1, T_{n-2} T_{n-3} \dots T_2 T_1 x_1^1), \\ \dots, d_1(x^1, T_n T_{n-1} \dots T_2 T_1 x_1^1) d_3(T_2 T_1 x_2^1, T_2 T_1 x_1^1), d_1(x^1, T_n T_{n-1} \dots T_2 T_1 x_1^1) \\ \left. d_2(T_1 x_2^1, T_1 T_n \dots T_3 T_2 T_1 x_2^1), d_1(x^1, T_n T_{n-1} \dots T_2 T_1 x_2^1) d_2(T_1 x_2^1, T_1 x_1^1) \right\}$$

$$F_2(x^2, x_2^2) = \max \left\{ d_2(x^2, T_1 T_n \dots T_3 T_2 x_2^2) d_1(T_n T_{n-1} \dots T_3 T_2 x_2^2, T_n T_{n-1} \dots T_3 T_2 x_2^2), d_2(x^2, T_1 T_n \dots T_3 T_2 x_2^2) \right. \\ d_n(T_{n-1} T_{n-2} \dots T_4 T_3 x_2^2, T_{n-1} T_{n-2} \dots T_3 T_2 x_2^2), d_2(x^2, T_1 T_n \dots T_3 T_2 x_2^2) \\ d_{n-1}(T_{n-2} T_{n-3} \dots T_3 T_2 x_2^2, T_{n-2} T_{n-3} \dots T_3 T_2 x_2^2), \dots, d_2(x^2, T_1 T_n \dots T_3 T_2 x_2^2) \\ \left. d_3(T_2 x_2^2, T_2 T_1 T_n \dots T_3 T_2 x_2^2), d_2(x^2, T_1 T_n \dots T_3 T_2 x_2^2) d_3(T_2 x_2^2, T_2 x_2^2) \right\}$$

$$F_3(x^3, x_2^3) = \max \left\{ d_3(x^3, T_2 T_1 T_n \dots T_4 T_3 x_2^3) d_2(T_1 T_n \dots T_4 T_3 x_2^3, T_1 T_n \dots T_4 T_3 x_2^3), d_3(x^3, T_2 T_1 T_n \dots T_4 T_3 x_2^3) \right. \\ d_1(T_n T_{n-1} \dots T_4 T_3 x_2^3, T_n T_{n-1} \dots T_4 T_3 x_2^3), d_3(x^3, T_2 T_1 T_n \dots T_4 T_3 x_2^3) \\ d_n(T_{n-1} T_{n-2} \dots T_4 T_3 x_2^3, T_{n-1} T_{n-2} \dots T_4 T_3 x_2^3), d_3(x^3, T_2 T_1 T_n \dots T_4 T_3 x_2^3) \\ d_{n-1}(T_{n-2} T_{n-3} \dots T_4 T_3 x_2^3, T_{n-2} T_{n-3} \dots T_4 T_3 x_2^3), \dots, d_3(x^3, T_2 T_1 T_n \dots T_4 T_3 x_2^3) \\ \left. d_4(T_3 x_2^3, T_3 T_2 T_1 T_n \dots T_4 T_3 x_2^3), d_3(x^3, T_2 T_1 T_n \dots T_4 T_3 x_2^3) d_4(T_3 x_2^3, T_3 x_2^3) \right\} \\ \vdots \\ \vdots$$

$$F_n(x^n, x_2^n) = \max \left\{ d_n(x^n, T_{n-1} T_{n-2} \dots T_1 T_n x_2^n) d_{n-1}(T_{n-2} T_{n-3} \dots T_1 T_n x_2^n, T_{n-2} T_{n-3} \dots T_1 T_n x_2^n), \right. \\ d_n(x^n, T_{n-1} T_{n-2} \dots T_1 T_n x_2^n) d_{n-2}(T_{n-3} T_{n-4} \dots T_1 T_n x_2^n, T_{n-3} T_{n-4} \dots T_1 T_n x_2^n), \dots, \\ \left. d_n(x^n, T_{n-1} T_{n-2} \dots T_1 T_n x_2^n) d_1(T_n x_2^n, T_n T_{n-1} \dots T_2 T_1 x_1^1), d_n(x^n, T_{n-1} T_{n-2} \dots T_1 T_n x_2^n) d_1(T_n x_2^n, T_n x_2^n) \right\}$$

And

$$G_1(x^1, x_2^1) = \max \left\{ d_1(x^1, T_n T_{n-1} \dots T_2 T_1 x_2^1), d_1(x^1, T_n T_{n-1} \dots T_2 T_1 x^1), d_2(T_1 x^1, T_1 T_n \dots T_2 T_1 x_2^1) \right\}$$

$$G_2(x^2, x_2^2) = \max \left\{ d_2(x^2, T_1 T_n \dots T_3 T_2 x_2^2), d_2(x^2, T_1 T_n \dots T_3 T_2 x^2), d_3(T_2 x^2, T_2 T_1 T_n \dots T_3 T_2 x_2^2) \right\}$$

$$\vdots$$

$$G_n(x^n, x^1) = \max \left\{ d_n(x^n, T_{n-1} T_{n-2} \dots T_1 T_n x^n), d_n(x^n, T_{n-1} T_{n-2} \dots T_1 T_n x^1), d_1(T_n x^n, T_n T_{n-1} \dots T_1 T_n x_1^n) \right\}$$

Then $T_n T_{n-1} \dots T_2 T_1$ has a unique fixed point $\alpha_1 \in X_1$, $T_1 T_n \dots T_3 T_2$ has a unique fixed point $\alpha_2 \in X_2$, and so on, $T_{n-1} T_{n-2} \dots T_1 T_n$ has a unique fixed point $\alpha_n \in X_n$.

$$\text{Further, } T_1(\alpha_1) = \alpha_2, T_2(\alpha_2) = \alpha_3, T_3(\alpha_3) = \alpha_4, \dots, T_{n-1}(\alpha_{n-1}) = \alpha_n, T(\alpha_n) = \alpha_1$$

Proof: Let $x'_0 \in X$, be any arbitrary point. We define sequences $\{x_m^1\}$, $\{x_m^2\}$, $\{x_m^3\}, \dots, \{x_m^n\}$ with $X_1, X_2, X_3, \dots, X_n$ respectively as follows:

$$x_m^1 = (T_n T_{n-1} \dots T_2 T_1)^m x_0^1, x_m^2 = T_1 x_{m-1}^1, x_m^3 = T_2(x_m^2), x_m^4 = T_3(x_m^3) \forall m \in \mathbb{N}$$

We assume that $x_m^1 \neq x_{m+1}^1, x_m^2 \neq x_{m+1}^2, x_m^3 \neq x_{m+1}^3, \dots, x_m^n \neq x_{m+1}^n \forall m \in \mathbb{N}$

Taking $x^2 = x_m^2$ and $x_2^2 = x_{m-1}^2$ in inequality (3.2), we get,

$$d_2(x_m^2, x_{m+1}^2) = d_2(T_1 T_n \dots T_3 T_2 x_{m-1}^2, T_1 T_n T_{n-1} \dots T_3 T_2(x_m^2))$$

$$\leq \frac{cF_2(x_m^2, x_{m-1}^2)}{G_2(x_m^2, x_{m-1}^2)}$$

$$c \max \left\{ d_2(x_m^2, T_1 T_n \dots T_3 T_2 x_m^2), d_1(T_n T_{n-1} \dots T_3 T_2 x_{m-1}^2, T_n T_{n-1} \dots T_2 x_m^2), d_2(x_m^2, T_1 T_n \dots T_3 T_2 x_m^2) \right.$$

$$d_n(T_{n-1} T_{n-2} \dots T_3 T_2 x_{m-1}^2, T_{n-1} \dots T_3 T_2 x_m^2), d_2(x_m^2, T_1 T_n \dots T_3 T_2 x_m^2), d_{n-1}(T_{n-2} T_{n-3} \dots T_3 T_2 x_{m-1}^2, T_{n-2} \dots T_3 T_2 x_m^2), \dots,$$

$$\left. d_2(x_m^2, T_1 T_n \dots T_3 T_2 x_m^2), d_3(T_2 x_{m-1}^2, T_2 T_1 T_n \dots T_2 x_{m-1}^2), d_2(x_m^2, T_1 T_n \dots T_3 T_2 x_{m-1}^2), d_3(T_1 x_{m-1}^2, T_2 x_m^2) \right\}$$

$$= \frac{\quad}{\max \left\{ d_2(x_m^2, x_m^2), d_2(x_m^2, x_{m+1}^2), d_3(x_m^3, x_m^3) \right\}}$$

$$d_2(x_m^2, x_{m+1}^2) \leq c \max \left\{ d_1(x_{m-1}^1, x_m^1), d_n(x_{m-1}^n, x_m^n), d_{n-1}(x_{m-1}^{n-1}, x_m^{n-1}), \dots, d_3(x_{m-1}^3, x_m^3) \right\} \tag{3.5}$$

Now taking $x^3 = x_m^3$ and $x_2^3 = x_{m-1}^3$ in inequality (3.3), we obtain,

$$d_3(x_m^3, x_{m+1}^3) = d_3(T_2 T_1 T_n \dots T_4 T_3 x_{m-1}^3, T_2 T_1 T_n \dots T_4 T_3 x_m^3) \leq \frac{cF_3(x_m^3, x_{m-1}^3)}{G_3(x_m^3, x_{m-1}^3)}$$

$$\begin{aligned}
 & c \max \left\{ d_3 \left(x_m^3, T_2 T_1 T_n \dots T_3 x_m^3 \right) d_2 \left(T_1 T_n \dots T_4 T_3 x_{m-1}^3, T_1 T_n \dots T_4 T_3 x_m^3 \right), d_3 \left(x_m^3, T_2 T_1 T_n \dots T_3 x_m^3 \right) \right. \\
 & d_1 \left(T_n \dots T_4 T_3 x_{m-1}^3, T_n \dots T_4 T_3 x_m^3 \right), d_3 \left(x_m^3, T_2 T_1 T_n \dots T_4 T_3 x_m^3 \right) d_{n-1} \left(T_{n-2} \dots T_4 T_3 x_{m-1}^3, T_{n-2} \dots T_4 T_3 x_m^3 \right), \dots, \\
 & \left. d_3 \left(x_m^3, T_2 T_1 T_n \dots T_4 T_3 x_m^3 \right) d_4 \left(T_3 x_{m-1}^3, T_3 T_2 T_1 T_n \dots T_4 T_3 x_m^3 \right), d_3 \left(x_m^3, T_2 T_1 T_3 \dots T_4 T_3 x_{m-1}^3 \right) d_4 \left(T_3 x_{m-1}^3, T_3 x_m^3 \right) \right\} \\
 & = \frac{\quad}{\left(\max \left\{ d_3 \left(x_m^3, x_m^3 \right), d_3 \left(x_m^3, x_{m+1}^3 \right), d_4 \left(x_m^4, x_m^4 \right) \right\} \right)} \\
 & = c \max \left\{ d_2 \left(x_m^2, x_{m+1}^2 \right), d_1 \left(x_{m-1}^1, x_m^1 \right), d_n \left(x_{m-1}^n, x_m^n \right), d_{n-1} \left(x_{m-1}^{n-1}, x_m^{n-1} \right), \dots, d_4 \left(x_{m-1}^4, x_m^4 \right) \right\}
 \end{aligned}$$

From inequality (3.5), we get,

$$\begin{aligned}
 d_3 \left(x_m^3, x_{m+1}^3 \right) \leq & \left\{ c d_1 \left(x_{m-1}^1, x_m^1 \right), c d_n \left(x_{m-1}^n, x_m^n \right), c d_{n-1} \left(x_{m-1}^{n-1}, x_m^{n-1} \right), \dots, c d_3 \left(x_{m-1}^3, x_m^3 \right), d_1 \left(x_{m-1}^1, x_m^1 \right) \right. \\
 & \left. , d_n \left(x_{m-1}^n, x_m^n \right), d_{n-1} \left(x_{m-1}^{n-1}, x_m^{n-1} \right), \dots, d_4 \left(x_{m-1}^4, x_m^4 \right) \right\}
 \end{aligned}$$

Since $0 \leq c < 1$, so, we get,

$$d_3 \left(x_m^3, x_{m+1}^3 \right) \leq c \max \left\{ d_1 \left(x_{m-1}^1, x_m^1 \right), d_n \left(x_{m-1}^n, x_m^n \right), d_{n-1} \left(x_{m-1}^{n-1}, x_m^{n-1} \right), \dots, d_4 \left(x_{m-1}^4, x_m^4 \right), d_3 \left(x_{m-1}^3, x_m^3 \right) \right\} \quad (3.6)$$

Similarly, we can get,

$$\begin{aligned}
 d_{n-1} \left(x_m^{n-1}, x_{m+1}^{n-1} \right) \leq & c \max \left\{ d_1 \left(x_{m-1}^1, x_m^1 \right), d_3 \left(x_{m-1}^3, x_m^3 \right), d_4 \left(x_{m-1}^4, x_m^4 \right), \dots, \right. \\
 & \left. d_{n-1} \left(x_{m-1}^{n-1}, x_m^{n-1} \right), d_n \left(x_{m-1}^n, x_m^n \right) \right\} \quad (3.7)
 \end{aligned}$$

Taking $x_2^n = x_{m-1}^n$ and $x^n = x_m^n$ in inequality (3.4)

$$\begin{aligned}
 d_n \left(x_m^n, x_{m+1}^n \right) & = d_n \left(T_{n-1} T_{n-2} \dots T_1 T_n x_{m-1}^n, T_{n-1} T_{n-2} \dots T_1 T_n x_m^n \right) \leq \frac{c F_n \left(x_m^n, x_{m-1}^n \right)}{G_n \left(x_m^n, x_{m-1}^n \right)} \\
 & c \max \left\{ d_n \left(x_m^n, T_{n-1} T_{n-2} \dots T_2 T_1 T_n x_m^n \right) d_{n-1} \left(T_{n-2} \dots T_1 T_n x_{m-1}^n, T_{n-2} \dots T_1 T_n x_m^n \right), \right. \\
 & d_n \left(x_m^n, T_{n-1} T_{n-2} \dots T_1 T_n x_m^n \right) d_{n-2} \left(T_{n-3} \dots T_1 T_n x_{m-1}^n, T_{n-3} \dots T_n x_m^n \right), \dots, d_n \left(x_m^n, T_{n-1} \dots T_1 T_n x_m^n \right) \\
 & \left. d_1 \left(T_n x_{m-1}^n, T_n T_{n-1} \dots T_1 T_n x_{m-1}^n \right), d_n \left(x_m^n, T_{n-1} \dots T_1 T_n x_{m-1}^n \right) d_1 \left(T_n x_{m-1}^n, T_n x_m^n \right) \right\} \\
 & = \frac{\quad}{\max \left\{ d_n \left(x_m^n, T_{n-1} T_{n-2} \dots T_1 T_n x_{m-1}^n \right), d_n \left(x_m^n, T_{n-1} T_{n-2} \dots T_1 T_n x_m^n \right), d_1 \left(T_n x_m^n, T_n T_{n-1} \dots T_2 T_1 T_n x_{m-1}^n \right) \right\}} \\
 & = c \max \left\{ d_{n-1} \left(x_m^{n-1}, x_{m+1}^{n-1} \right), d_{n-2} \left(x_m^{n-2}, x_{m+1}^{n-2} \right), \dots, d_1 \left(x_{m-1}^1, x_m^1 \right) \right\}
 \end{aligned}$$

From (3.5), (3.6) and (3.7), we get,

$$d_n \left(x_m^n, x_{m+1}^n \right) \leq c \max \left\{ d_1 \left(x_{m-1}^1, x_m^1 \right), d_3 \left(x_{m-1}^3, x_m^3 \right), d_4 \left(x_{m-1}^4, x_m^4 \right), \dots, d_{n-1} \left(x_{m-1}^{n-1}, x_m^{n-1} \right), d_n \left(x_{m-1}^n, x_m^n \right) \right\} \quad (3)$$

It now follows by induction on above inequalities, we get,

$$d_1 \left(x_m^1, x_{m+1}^1 \right) \leq c^{m-1} \max \left\{ d_1 \left(x_1^1, x_2^1 \right), d_3 \left(x_1^3, x_2^3 \right), \dots, d_{n-1} \left(x_1^{n-1}, x_2^{n-1} \right), d_n \left(x_1^n, x_2^n \right) \right\}$$

$$d_2 \left(x_m^2, x_{m+1}^2 \right) \leq c^{m-1} \max \left\{ d_1 \left(x_1^1, x_2^1 \right), d_3 \left(x_1^3, x_2^3 \right), \dots, d_{n-1} \left(x_1^{n-1}, x_2^{n-1} \right), d_n \left(x_1^n, x_2^n \right) \right\}$$

⋮

$$d_n(x_m^n, x_{m+1}^n) \leq c^{m-1} \max \left\{ d_1(x_1^1, x_2^1), d_3(x_1^3, x_2^3), \dots, d_{n-1}(x_1^{n-1}, x_2^{n-1}), d_n(x_1^n, x_2^n) \right\}$$

Since $0 \leq c < 1$, the sequence $\{x_m^1\}, \{x_m^2\}, \{x_m^3\}, \dots, \{x_m^{n-1}\}, \{x_m^n\}$ are Cauchy sequence in $X_1, X_2, X_3, \dots, X_n$. So we have,

$$\lim_{m \rightarrow \infty} x_m^i = \alpha_i \text{ when } i = 1, 2, 3, \dots, n.$$

Since $T_1, T_2, T_3, \dots, T_{n-1}$ are continuous mapping.

$$T_1\alpha_1 = \alpha_2, T_2\alpha_2 = \alpha_3, \dots, T_{n-1}\alpha_{n-1} = \alpha_n$$

Now taking $x^1 = x_{m-1}^1$ and $x^2 = \alpha_1$ in inequality (3.1),

$$d_1(T_n T_{n-1} \dots T_2 T_1 \alpha_1, T_n T_{n-1} \dots T_2 T_1 x_m^1) \leq \frac{c F_1(x_{m-1}^1, \alpha_1)}{G_1(x_{m-1}^1, \alpha_1)}$$

$$c \max \left\{ d_1(x_{m-1}^1, T_n \dots T_2 T_1 x_{m-1}^1) d_n(T_{n-1} \dots T_2 T_1 \alpha_1, T_{n-1} \dots T_2 T_1 x_{m-1}^1) d_1(x_{m-1}^1, T_n \dots T_2 T_1 x_{m-1}^1) \right.$$

$$d_{n-1}(T_{n-2} \dots T_2 T_1 \alpha_1, T_{n-2} \dots T_2 T_1 x_{m-1}^1), \dots, d_1(x_{m-1}^1, T_n \dots T_2 T_1 x_{m-1}^1) d_3(T_2 T_1 \alpha_1, T_2 T_1 x_{m-1}^1), d_1(x_{m-1}^1, T_n \dots T_2 T_1 x_{m-1}^1)$$

$$\left. d_2(T_1 \alpha_1, T_1 T_n \dots T_2 T_1 \alpha_1), d_1(x_{m-1}^1, T_1 T_n T_{n-1} \dots T_2 T_1 \alpha_1) d_2(T_1 \alpha_1, T_1 x_{m-1}^1) \right\}$$

$$= \frac{\max \left\{ d_1(x_{m-1}^1, T_n T_{n-1} \dots T_2 T_1 \alpha_1), d_1(x_{m-1}^1, T_n T_{n-1} \dots T_2 T_1, x_{m-1}^1), d_2(T_1 x_{m-1}^1, T_1 T_n \dots T_2 T_1 \alpha_1) \right\}}$$

As $m \rightarrow \infty$, we get,

$$d_1(T_n T_{n-1} \dots T_2 T_1 \alpha_1, \alpha_1) \leq \frac{c d_1(\alpha_1, T_1 T_n T_{n-1} \dots T_2 T_1 \alpha_1) d_2(T_1 \alpha_1, \alpha_2)}{\max \{ d_1(\alpha_1, T_n T_{n-1} \dots T_2 T_1 \alpha_1), d_2(\alpha_2, T_1 T_n \dots T_2 T_1 \alpha_1) \}}$$

Since T_1 is continuous mapping.

$$\Rightarrow d_1(T_n T_{n-1} \dots T_2 T_1 \alpha_1, \alpha_1) \leq 0$$

Since, which is not possible

$$\Rightarrow T_n T_{n-1} \dots T_2 T_1 \alpha_1 = \alpha_1 \tag{3.9}$$

$$\begin{aligned} \text{Now, } T_1 T_n T_{n-1} \dots T_3 T_2 \alpha_2 &= T_1 T_n T_{n-1} \dots T_3 T_2 (T_1 \alpha_1) = T_1 (T_n T_{n-1} \dots T_3 T_2 T_1 \alpha_1) \\ &= T_1 \alpha_1 = \alpha_2 \\ \Rightarrow T_1 T_n T_{n-1} \dots T_3 T_2 \alpha_2 &= \alpha_2 \end{aligned}$$

Since $T_1, T_2, T_3, \dots, T_{n-1}$ are continuous mapping, So in same above way, so we obtain

$$\begin{aligned} T_1 T_n T_{n-1} \dots T_2 \alpha_2 &= \alpha_2 \\ T_2 T_1 T_n \dots T_3 \alpha_3 &= \alpha_3 \\ T_3 T_2 T_1 \dots T_4 \alpha_4 &= \alpha_4 \\ &\vdots \\ T_{n-1} \dots T_2 T_1 T_n \alpha_n &= \alpha_n \end{aligned}$$

and $\alpha_1 \in X_1, \alpha_2 \in X_2, \dots, \alpha_n \in X_n$ and $T_1\alpha_1 = \alpha_2, T_2\alpha_2 = \alpha_3, \dots, T_{n-1}\alpha_{n-1} = \alpha_n$.

Uniqueness

Let us assume that α'_1 is another fixed point of $T_n T_{n-1} \dots T_2 T_1$ different from α_1 such that $\alpha'_1 \neq \alpha_1$ and

$$T_n T_{n-1} \dots T_2 T_1 \alpha'_1 = \alpha'_1, T_n T_{n-1} \dots T_2 T_1 \alpha_1 = \alpha_1.$$

Now taking $x^1 = \alpha'_1$ and $x^1 = \alpha_1$ in inequality (3.1), we get,

$$\begin{aligned} d_1(\alpha, \alpha'_1) &= d_1(T_n T_{n-1} \dots T_2 T_1 \alpha_1, T_n T_{n-1} \dots T_1 \alpha'_1) \\ &\leq \frac{cF_1(\alpha'_1, \alpha_1)}{G_1(\alpha'_1, \alpha_1)} \\ &= \frac{\left(\begin{aligned} &c \max \{d_1(\alpha'_1, T_n \dots T_2 T_1 \alpha'_1) d_n(T_{n-1} \dots T_3 T_2 T_1, T_{n-1} \dots T_2 T_1 \alpha'_1), d_1(\alpha'_1, T_n \dots T_2 T_1 \alpha'_1) \\ &d_{n-1}(T_{n-2} \dots T_3 T_2 T_1 \alpha_1, T_{n-2} \dots T_2 T_1 \alpha'_1), \dots, d_1(\alpha'_1, T_n \dots T_2 T_1 \alpha'_1) d_3(T_2 T_1 \alpha_1, T_2 T_1 \alpha'_1), \\ &d_1(\alpha'_1, T_n \dots T_2 T_1 \alpha'_1) d_2(T_1 \alpha_1, T_1 T_n \dots T_3 T_2 T_1 \alpha_1), d_1(\alpha'_1, T_n T_{n-1} \dots T_3 T_2 T_1 \alpha_1) d_2(T_1 \alpha_1, T_1 \alpha'_1)\} \end{aligned} \right)}{\left(\max \{d_1(\alpha'_1, T_n T_{n-1} \dots T_3 T_2 T_1 \alpha_1), d_1(\alpha'_1, T_n T_{n-1} \dots T_2 T_1 \alpha'_1), d_2(T_1 \alpha'_1, T_1 T_n \dots T_2 T_1 \alpha_1)\} \right)} \\ &= \frac{cd_1(\alpha'_1, \alpha_1) d_2(T_1 \alpha, T_1 \alpha'_1)}{\max \{d_1(\alpha'_1, \alpha_1), d_2(T_1 \alpha'_1, T_1 \alpha_1)\}} \\ &\Rightarrow d_1(\alpha_1, \alpha'_1) \leq cd_2(T_1 \alpha_1, T_1 \alpha'_1) \end{aligned} \tag{3.10}$$

In the same way, taking $x^2 = T_1 \alpha_1$ and $x^2 = T_1 \alpha'_1$ in inequality (3.2), we get

$$\begin{aligned} d_2(T_1 T_n T_{n-1} \dots T_3 T_2 T_1 \alpha_1, T_1 T_n T_{n-1} \dots T_2 T_1 \alpha'_1) &\leq \frac{cF_2(T_1 \alpha'_1, T_1 \alpha_1)}{G_2(T_1 \alpha'_1, T_1 \alpha_1)} \\ &= \frac{\left(\begin{aligned} &c \max \{d_2(T_1 \alpha'_1, T_1 \alpha'_1) d_1(T_n T_{n-1} \dots T_2 T_1 \alpha_1, \\ &T_n \dots T_2 T_1 \alpha'_1), \dots, d_2(T_1 \alpha'_1, T_1 \alpha_1) d_3(T_2 T_1 \alpha_1, T_2 T_1 \alpha'_1)\} \end{aligned} \right)}{\left(\max \{d_2(T_1 \alpha'_1, T_1 \alpha_1), d_2(T_1 \alpha'_1, T_1 \alpha'_1), d_3(T_2 T_1 \alpha'_1, T_2 T_1 \alpha_1)\} \right)} \\ &\Rightarrow d_2(T_1 \alpha_1, T_1 \alpha'_1) \leq cd_3(T_2 T_1 \alpha_1, T_2 T_1 \alpha'_1) \end{aligned} \tag{3.11}$$

Similarly, by taking $x^3 = T_2 T_1 \alpha_1$ and $x^3 = T_2 T_1 \alpha'_1$ in inequality (3.3), we get,

$$d_3(T_2 T_1 \alpha_1, T_2 T_1 \alpha'_1) \leq cd_4(T_3 T_2 T_1 \alpha_1, T_3 T_2 T_1 \alpha'_1) \tag{3.12}$$

Continue like this, taking $x^n = T_{n-1} T_{n-2} \dots T_2 T_1 \alpha_1$ and $x^n = T_{n-1} \dots T_2 T_1 \alpha'_1$ in inequality (3.4), we get,

$$\begin{aligned} d_n(T_{n-1} T_{n-2} \dots T_2 T_1 \alpha_1, T_{n-1} \dots T_2 T_1 \alpha'_1) &\leq cd_1(T_n T_{n-1} \dots T_2 T_1 \alpha_1, T_n T_{n-1} \dots T_2 T_1 \alpha'_1) \\ &= cd_1(\alpha_1, \alpha'_1) \end{aligned} \tag{3.13}$$

From inequalities (3.10),(3.11), (3.12)and(3.13), we get,

$$\begin{aligned} d_1(\alpha_1, \alpha'_1) &\leq cd_2(T_1\alpha_1, T_1\alpha'_1) \\ &\leq c^2d_3(T_2T_1\alpha_1, T_2T_1\alpha'_1) \\ &\leq c^3d_4(T_3T_2T_1\alpha_1, T_3T_2\alpha'_1) \\ &\vdots \\ &\leq c^{n-1}d_n(T_{n-1}\dots T_2T_1\alpha_1, T_{n-1}\dots T_2T_1\alpha'_1) \\ &\leq c^nd_1(\alpha_1, \alpha'_1) \end{aligned}$$

$$\Rightarrow d_1(\alpha_1, \alpha'_1) \leq c^nd_1(\alpha_1, \alpha'_1)$$

Since $0 \leq c < 1$, so we get,

$$\begin{aligned} &\Rightarrow d_1(\alpha_1, \alpha'_1) = 0 \\ &\Leftrightarrow \alpha_1 = \alpha'_1 \end{aligned}$$

So, α_1 is unique fixed point of $T_nT_{n-1}\dots T_2T_1$.

In the similar way, the unicity of other fixed point can be proved.

We have to finally proved that $T(\alpha_n) = \alpha_1$, To do this, note that

$$T_n(\alpha_n) = T_n(T_{n-1}\dots T_2T_1T_n\alpha_n) = T_nT_{n-1}\dots T_2T_1(T_n\alpha_n)$$

And so $T_n\alpha_n$ is fixed point of $T_nT_{n-1}\dots T_2T_1$. Since α_1 is the unique fixed point of $T_nT_{n-1}\dots T_2T_1$. It follows that $T(\alpha_n) = \alpha_1$

So, we obtain $T_nT_{n-1}\dots T_2T_1$ has a unique fixed point $\alpha_1 \in X_1$, $T_1T_n\dots T_3T_2$ has a unique fixed point $\alpha_2 \in X_2$, and so on, $T_{n-1}T_{n-2}\dots T_1T_n$ has a unique fixed point $\alpha_n \in X_n$ Also $T_1(\alpha_1) = \alpha_2, T_2(\alpha_2) = \alpha_3, T_3(\alpha_3) = \alpha_4, \dots, T_{n-1}(\alpha_{n-1}) = \alpha_n, T(\alpha_n) = \alpha_1$

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