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A NOTE ON QUASI METRIC SPACES

P. SUMATI KUMARI*

Department of Mathematics, FED – I, K L University, Green fields, Vaddeswaram, A.P, 522502, India
E-mail: mumy143143143@gmail.com

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ABSTRACT

Quasi metrics have been used in several places in the literature on domain theory and the formal semantics of programming languages [1],[3]. The purpose of this paper is to establish some fixed point theorems in quasi metric spaces.

Keywords: *Quasi metric, fixed point.*

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Section-1

We denote the set of non-negative real numbers by R^+ and set of natural numbers by N.

Definition 1.0:

Let X be a set and $d: X \times X \to R^+$ be a function, satisfying following conditions

I. $d(x,x)=0$

II. $d(x,y)= d(y, x)=0$ implies $x=y$

III. $d(x,y) \le d(x,z)+d(z, y)$ for all $x, y, z \in X$

Then d is called a quasi metric on X .

Definition 1.1 [1]: A sequence (x_n) in a quasi metric space (X ,d) is a (forward) Cauchy sequence if, for all $\in >0$, there corresponds $n_\varepsilon \in N$ such that for all $n \ge m \ge n_\varepsilon$ we have $d(x_m, x_n) < \in$. A Cauchy sequence (x_n) converges to $x \in X$ if, for all $y \in X$, $d(x,y)=\lim d(x_n, y)$. In this case we write $\lim x_n = x$. Finally X is called CS-complete if every Cauchy sequence in X converges.

Note [1]: limits of Cauchy sequence in quasi metric spaces are unique.

Definition 1.2: Let X be a quasi metric space. A function $f: X \to X$ is called

(1) CS-continuous if, for all Cauchy sequences (x_n) in X with $\lim x_n = x$, ($f(x_n)$) is a Cauchy sequence and $\lim f(x_n) = f(x)$.

(2) Non expanding if $d(f(x), f(y)) \le d(x,y)$ for all $x, y \in X$.

Theorem 1.3: (Theorem 1.6.3[1]) Let (X ,d) be a CS-complete quasi metric space. And let $f: X \to X$ be non expanding. If f is CS-continuous and contractive, then f has a unique fixed point.

Theorem 1.4: Let (X ,d) be a CS-complete quasi metric space and $f: X \to X$ be CS-continuous. Assume that there exist non-negative constants a_i satisfying $a_1 + a_2 + a_3 + 2a_4 < 1$ such that for each $x, y \in X \; with \; x \ne y$

***Corresponding author:** P. SUMATI KUMARI*, ***E-mail:** mumy143143143@gmail.com*

$$d\big(f(x), f(y)\big) \le a_1 d(x,y) + a_2 d(x, f(x)) + a_3 d(y, f(y)) + a_4 d(x, f(y)) + a_5 d(y, f(x))$$

Then 'f' has a unique fixed point.

Proof: For any $x \in X$

$$d\big(f(x), f^2(x)\big) \le a_1 d(x, f(x)) + a_2 d(x, f(x)) + a_3 d(f(x), f^2(x)) + a_4 d(x, f^2(x)) + a_5 d(f(x), f^2(x))$$

$$\le (a_1 + a_2 + a_4) d(x, f(x)) + (a_3 + a_4) d(f(x), f^2(x))$$

$\Rightarrow d(f(x), f^2(x)) \le \beta \ d(x, f(x))$ Where $\beta = \dfrac{a_1 + a_2 + a_4}{1 - a_3 - a_4}$. *clearly* $0 \le \beta < 1$

If m > n then,

$$d(f^n(x), f^m(x)) \le \beta^n \left(1 + \beta + \ldots\ldots + \beta^{m-n-1}\right) d(x, f(x)) \le \frac{\beta^n}{1-\beta} d(x, f(x))$$

Hence { $f^n(x)$ } is Cauchy's sequence in (X, d), hence convergent.

Let $\xi = \lim\limits_n f^n(x)$ Then $f^{n+1}(x)$ is Cauchy and $f(\xi) = \lim\limits_n f^{n+1}(x)$

$$d(\xi, f(\xi)) = \lim_n d(f^n(x) f(\xi)) = \lim d(f^{n+1}(x) f(\xi)) = 0$$

$$d(f(\xi), \xi) = \lim d(f^n(x), f(\xi)) = \lim d(f^{n+1}(x) f(\xi)) = 0$$

$\Rightarrow f(\xi) = \xi$

UNIQUENESS:

Suppose $f(\xi) = \xi$ *and* $f(\eta) = \eta$

$$d(\xi, \eta) = \lim d(f^n(x), \eta) = \lim d(f^{n+1}(x), \eta) = 0$$

Similarly, $d(\xi, \eta) = 0$ *Hence* $\xi = \eta$

Theorem 1.5: Let (X , d) be a CS-complete quasi metric space and $f : X \to X$ be a CS-continuous mapping, there exist real numbers $\alpha, \beta, \gamma \ni, 0 \le \alpha < \frac{1}{2}$, $0 \le \beta < \frac{1}{2}$, $\gamma < \min\{\frac{1}{4}, \frac{1}{2} - \alpha, \frac{1}{2} - \beta\}$ and

For each $x, y \in X$ at least one of the following holds

i. $d(f(x), f(y)) \le \alpha d(x, y)$
ii. $d(f(x), f(y)) \le \beta \, [d(x, f(x)) + d(y, f(y))]$
iii. $d(f(x), f(y)) \le \gamma \, [d(x, f(y)) + d(y, f(x))]$

Then f has a fixed point.

Proof: put $y = x$ in the above.

(i) $d(f(x), f^2(x)) \le \alpha \ d(x, f(x))$

(ii) $d(f(x), f^2(x)) \le \beta \, [d(x, f(x)) + d(f(x), f^2(x))]$

$$\Rightarrow d(f(x), f^2(x)) \le \frac{\beta}{1-\beta} d(x, f(x))$$

(iii) in this case

$$d(f(x), f^2(x)) \le \frac{3\gamma}{1-\gamma} d(x, f(x))$$

If h = max { α , $\frac{\beta}{1-\beta}$, $\frac{\gamma}{1-\gamma}$ } then $0 \le h < 1$ and $d(f(x), f^2(x)) \le h\ d(x, f(x))$

Hence $\{f^n(x)\}$ is a Cauchy sequence.

Since X is CS-Complete, there exists 'z ' such that

$\lim_n f^n(x) = z$ in (X , d)

Hence $\lim_n f^{n+1}(x) = f(z)$ in (X , d)

$\because\ 0 \le d(z, f(z)) \le d(z, f^{n+1}(x)) + d(f^{n+1}(x), f(z))$

$d(z, f(z))$=0 similarly $d(f(z), z) = 0$

Heance $f(z) = z$ Which proves the theorem.

Section-2

B.E Rhodes [4] presented a list of definitions of contractive type conditions for a self map on a metric space (X , d) and established implications and non implications among them ,there by facilitating to check the implication of any new contractive condition through any one of the condition mentioned in [4] so as to derive a fixed point theorem. We now present the quasi metric versions some of them.

Let (X , d) be a quasi metric space and $f : X \to X$ be a mapping and x, y be any elements of X . Consider the following conditions.

1. (Banach): there exists a number a , 0 $\le a \le 1$ such that $d(f(x), (y)) \le$ a $d(x, y)$
2. (Rakotch): there exists a monotone decreasing function $\alpha : [0, \infty) \to [0,1)$ such that

$$d(f(x), (y)) \le \alpha\ d(x, y) \quad \text{whenever } d(x, y) \ne 0$$

3. (Edelstein): $d(f(x), f(y)) < d(x, y)$ whenever $d(x, y) \ne 0$
4. (Kannan) :there exists a number a, $0 < a < \frac{1}{2}$ such that

$$d(f(x), f(y)) < \text{a}\,[d(x, f(x)) + d(y, f(y))]$$

5. (Bianchini): there exists a number h, 0 $\le h < 1$ such that

$$d(f(x), f(y)) \le \text{h max}\{ d(x, f(x)), d(y, f(y)) \}$$

6. (Reich): there exist nonnegative numbers a, b, c satisfying a + b + c < 1 such that

$$d(f(x), f(y)) \le \text{a } d(x, f(x)) + \text{b } d(y, f(y)) + \text{c } d(x, y)$$

7. (Reich): there exist monotonically decreasing functions a, b, c from (0 , ∞) to [0 ,1) satisfying a(t) + b(t) + c(t) < 1 such that ,

$$d(f(x), f(y)) < \text{a(t) } d(x, f(x)) + \text{b(t) } d(y, f(y)) + \text{c(t)t} \quad \text{where t=} d(x, y)$$

8. $d(f(x), f(y)) \le a(x,y)\ d(x, f(y)) + b(x,y)\ d(y, f(x)) + c(x,y)\ d(x, y)$,

$$\sup_{x,y \in X} \{a(x,y) + b(x,y) + c(x,y)\} \le \lambda < 1$$

9. (Ciric): For each $x, y \in X$

$$d(f(x), f(y)) \leq q(x,y)\, d(x, y) + r(x,y)\, d(x, f(x)) + s(x,y)\, d(y, f(y)) + t(x,y)\,[d(x, f(y)) + d(y, f(x))]$$

$$\sup_{x,y \in X} \{q(x,y) + r(x,y) + s(x,y) + 2t(x,y) \leq \lambda < 1$$

Theorem 2.1: Let (X , d) be a CS-complete quasi metric space and let $f : X \to X$ be non expanding. If f satisfies one of the above mentioned conditions, then f has a unique fixed point.

Proof: It now follows from Theorem 1.4 that f has a fixed point.

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