



A NOTE ON QUASI METRIC SPACES

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ABSTRACT

Quasi metrics have been used in several places in the literature on domain theory and the formal semantics of programming languages [1],[3]. The purpose of this paper is to establish some fixed point theorems in quasi metric spaces.

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Section-1

We denote the set of non-negative real numbers by R^+ and set of natural numbers by N .

Definition 1.0:

Let X be a set and $d : X \times X \rightarrow R^+$ be a function, satisfying following conditions

- I. $d(x, x) = 0$
- II. $d(x, y) = d(y, x) = 0$ implies $x = y$
- III. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

Then d is called a quasi metric on X .

Definition 1.1 [1]: A sequence (x_n) in a quasi metric space (X, d) is a (forward) Cauchy sequence if, for all $\epsilon > 0$, there corresponds $n_\epsilon \in N$ such that for all $n \geq m \geq n_\epsilon$ we have $d(x_m, x_n) < \epsilon$. A Cauchy sequence (x_n) converges to $x \in X$ if, for all $y \in X$, $d(x, y) = \lim d(x_n, y)$. In this case we write $\lim x_n = x$. Finally X is called CS-complete if every Cauchy sequence in X converges.

Note [1]: limits of Cauchy sequence in quasi metric spaces are unique.

Definition 1.2: Let X be a quasi metric space. A function $f : X \rightarrow X$ is called

- (1) CS-continuous if, for all Cauchy sequences (x_n) in X with $\lim x_n = x$, $(f(x_n))$ is a Cauchy sequence and $\lim f(x_n) = f(x)$.
- (2) Non expanding if $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$.

Theorem 1.3: (Theorem 1.6.3[1]) Let (X, d) be a CS-complete quasi metric space. And let $f : X \rightarrow X$ be non expanding. If f is CS-continuous and contractive, then f has a unique fixed point.

Theorem 1.4: Let (X, d) be a CS-complete quasi metric space and $f : X \rightarrow X$ be

CS-continuous. Assume that there exist non-negative constants a_i satisfying $a_1 + a_2 + a_3 + 2a_4 < 1$ such that for each $x, y \in X$ with $x \neq y$

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$$d(f(x), f(y)) \leq a_1 d(x, y) + a_2 d(x, f(x)) + a_3 d(y, f(y)) + a_4 d(x, f(y)) + a_5 d(y, f(x))$$

Then 'f' has a unique fixed point.

Proof: For any $x \in X$

$$\begin{aligned} d(f(x), f^2(x)) &\leq a_1 d(x, f(x)) + a_2 d(x, f(x)) + a_3 d(f(x), f^2(x)) + a_4 d(x, f^2(x)) + a_5 d(f(x), f^2(x)) \\ &\leq (a_1 + a_2 + a_4) d(x, f(x)) + (a_3 + a_5) d(f(x), f^2(x)) \end{aligned}$$

$$\Rightarrow d(f(x), f^2(x)) \leq \beta d(x, f(x)) \quad \text{Where } \beta = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_5} \quad \text{clearly } 0 \leq \beta < 1$$

If $m > n$ then,

$$d(f^n(x), f^m(x)) \leq \beta^n (1 + \beta + \dots + \beta^{m-n-1}) d(x, f(x)) \leq \frac{\beta^n}{1 - \beta} d(x, f(x))$$

Hence $\{f^n(x)\}$ is Cauchy's sequence in (X, d) , hence convergent.

Let $\xi = \lim_n f^n(x)$ Then $f^{n+1}(x)$ is Cauchy and $f(\xi) = \lim_n f^{n+1}(x)$

$$d(\xi, f(\xi)) = \lim_n d(f^n(x), f(\xi)) = \lim_n d(f^{n+1}(x), f(\xi)) = 0$$

$$d(f(\xi), \xi) = \lim_n d(f^n(x), f(\xi)) = \lim_n d(f^{n+1}(x), f(\xi)) = 0$$

$$\Rightarrow f(\xi) = \xi$$

UNIQUENESS:

Suppose $f(\xi) = \xi$ and $f(\eta) = \eta$

$$d(\xi, \eta) = \lim_n d(f^n(x), \eta) = \lim_n d(f^{n+1}(x), \eta) = 0$$

Similarly, $d(\xi, \eta) = 0$ Hence $\xi = \eta$

Theorem 1.5: Let (X, d) be a CS-complete quasi metric space and $f : X \rightarrow X$ be a CS-continuous mapping, there exist real numbers α, β, γ \ni $0 \leq \alpha < \frac{1}{2}$, $0 \leq \beta < \frac{1}{2}$, $\gamma < \min\{\frac{1}{4}, \frac{1}{2} - \alpha, \frac{1}{2} - \beta\}$ and

For each $x, y \in X$ at least one of the following holds

- i. $d(f(x), f(y)) \leq \alpha d(x, y)$
- ii. $d(f(x), f(y)) \leq \beta [d(x, f(x)) + d(y, f(y))]$
- iii. $d(f(x), f(y)) \leq \gamma [d(x, f(y)) + d(y, f(x))]$

Then f has a fixed point.

Proof: put $y = x$ in the above.

$$(i) d(f(x), f^2(x)) \leq \alpha d(x, f(x))$$

$$(ii) d(f(x), f^2(x)) \leq \beta [d(x, f(x)) + d(f(x), f^2(x))]$$

$$\Rightarrow d(f(x), f^2(x)) \leq \frac{\beta}{1 - \beta} d(x, f(x))$$

(iii) in this case

$$d(f(x), f^2(x)) \leq \frac{3\gamma}{1 - \gamma} d(x, f(x))$$

If $h = \max \left\{ \alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma} \right\}$ then $0 \leq h < 1$ and $d(f(x), f^2(x)) \leq h d(x, f(x))$

Hence $\{f^n(x)\}$ is a Cauchy sequence.

Since X is CS-Complete, there exists 'z' such that

$$\lim_n f^n(x) = z \text{ in } (X, d)$$

$$\text{Hence } \lim_n f^{n+1}(x) = f(z) \text{ in } (X, d)$$

$$\because 0 \leq d(z, f(z)) \leq d(z, f^{n+1}(x)) + d(f^{n+1}(x), f(z))$$

$$d(z, f(z)) = 0 \text{ similarly } d(f(z), z) = 0$$

Hence $f(z) = z$ Which proves the theorem.

Section-2

B.E Rhodes [4] presented a list of definitions of contractive type conditions for a self map on a metric space (X, d) and established implications and non implications among them, there by facilitating to check the implication of any new contractive condition through any one of the condition mentioned in [4] so as to derive a fixed point theorem. We now present the quasi metric versions some of them.

Let (X, d) be a quasi metric space and $f : X \rightarrow X$ be a mapping and x, y be any elements of X . Consider the following conditions.

1. (Banach): there exists a number $a, 0 \leq a \leq 1$ such that $d(f(x), f(y)) \leq a d(x, y)$
2. (Rakotch): there exists a monotone decreasing function $\alpha : [0, \infty) \rightarrow [0, 1]$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y) \text{ whenever } d(x, y) \neq 0$$

3. (Edelstein): $d(f(x), f(y)) < d(x, y)$ whenever $d(x, y) \neq 0$

4. (Kannan) :there exists a number $a, 0 < a < \frac{1}{2}$ such that

$$d(f(x), f(y)) < a [d(x, f(x)) + d(y, f(y))]$$

5. (Bianchini): there exists a number $h, 0 \leq h < 1$ such that

$$d(f(x), f(y)) \leq h \max\{d(x, f(x)), d(y, f(y))\}$$

6. (Reich): there exist nonnegative numbers a, b, c satisfying $a + b + c < 1$ such that

$$d(f(x), f(y)) \leq a d(x, f(x)) + b d(y, f(y)) + c d(x, y)$$

7. (Reich): there exist monotonically decreasing functions a, b, c from $(0, \infty)$ to $[0, 1]$ satisfying $a(t) + b(t) + c(t) < 1$ such that ,

$$d(f(x), f(y)) < a(t) d(x, f(x)) + b(t) d(y, f(y)) + c(t)t \text{ where } t = d(x, y)$$

8. $d(f(x), f(y)) \leq a(x, y) d(x, f(y)) + b(x, y) d(y, f(x)) + c(x, y) d(x, y),$

$$\sup_{x, y \in X} \{a(x, y) + b(x, y) + c(x, y)\} \leq \lambda < 1$$

9. (Ciric): For each $x, y \in X$

$$d(f(x), f(y)) \leq q(x, y) d(x, y) + r(x, y) d(x, f(x)) + s(x, y) d(y, f(y)) + t(x, y) [d(x, f(y)) + d(y, f(x))] \\ \sup_{x, y \in X} \{q(x, y) + r(x, y) + s(x, y) + 2t(x, y)\} \leq \lambda < 1$$

Theorem 2.1: Let (X, d) be a CS-complete quasi metric space and let $f : X \rightarrow X$ be non expanding. If f satisfies one of the above mentioned conditions, then f has a unique fixed point.

Proof: It now follows from Theorem 1.4 that f has a fixed point.

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