International Journal of Mathematical Archive-3(3), 2012, Page: 855-863
Available online through www.ijma.info ISSN 2229 – 5046

αg-separation axioms

S. Balasubramanian[1*] & Ch. Chaitanya[2]

[1]Department of Mathematics, Government Arts College (A), Karur – 639 005 Tamilnadu, India

[2]Sri Mittapalli college of Engineering, Tummalapalem, Guntur, Andhrapradesh, India

E-mail: mani55682@rediffmail.com [1] & chspurthi@yahoo.com [2]

(Received on: 08-12-11; Accepted on: 31-12-11)

Abstract

In this paper we discuss new separation axioms using αg-open sets.

Mathematics Subject Classification Number: *54D10, 54D15.*

***Keywords**: αg_i spaces*

---

1. Introduction:

Norman Levine introduced generalized closed sets in 1970. After him various Authors studied different versions of generalized sets and related topological properties. Recently V.K. Sharma and the author of the present paper defined separation axioms for g-open; gs-open; sg-open and rg-open sets.

Definition 1.1: $A \subseteq X$ is called generalized closed[resp: regular generalized; generalized regular]{briefly: g-closed; rg-closed; αg-closed}if cl{A}⊆U whenever A⊆U and U is open [resp: regular open, open] and generalized [resp: regular generalized; generalized regular] open if its complement is generalized [resp: regular generalized; generalized regular] closed.

Note 1: The class of regular open sets, open sets, α-open sets, αg-open sets and rg-open are denoted by RO(X), τ(X), αO(X) and αGO(X) respectively. Clearly $RO(X) \subset \tau(X) \subset \alpha O(X) \subset \alpha GO(X)$.

Note 2: $A \in \alpha gO(X, x)$ means A is α-generalized open neighborhood of X containing x.

Definition 1.2: $A \subset X$ is called clopen[resp: αg-clopen] if it is both open[resp: αg-open] and closed[resp: αg-closed]

Definition 1.3: A function $f: X \to Y$ is said to be
(i) g-continuous [resp: αg-continuous] if inverse image of closed set is g-closed [resp: αg-closed] and g-irresolute [resp: αg-irresolute] if inverse image of g-closed [resp: αg-closed] set is g-closed [resp: αg-closed]
(ii) αg-open if the image of open set αg-open
(iii) αg-homeomorphism [resp: αgc-homeomorphism] if f is bijective, αg-continuous[resp: αg-irresolute] and f^{-1} is αg-continuous[resp: αg-irresolute]

Definition 1.4: X is said to be
(i) compact [resp: nearly compact, g-compact, αg-compact] if every open[resp: regular-open, g-open, αg-open] cover has a finite sub cover.
(ii) T_0 [resp: rT_0, g_0, αg_0] space if for each $x \neq y \in X$ $\exists$ $U \in \tau(X)$[resp: RO(X); GO(X); αGO(X)] containing either x or y.
(iii) T_1 [resp: rT_1, g_1, αg_1]{ T_2 [resp: rT_2, g_2, αg_2]} space if for each $x \neq y \in X$ $\exists$ {disjoint} U, $V \in \tau(X)$[resp: RO(X); GO(X); αGO(X)] αg-open sets G and H containing x and y respectively.
(iv) $T_{1/2}$ [resp: $rT_{1/2}$, $\alpha T_{1/2}$] if every generalized [resp: regular generalized, α-generalized] closed set is closed [resp: regular-closed, α-closed]

Corresponding author: S. Balasubramanian[1]*,*E-mail: mani55682@rediffmail.com

2. αg-continuity and product spaces:

Theorem 2.1: Let Y and $\{X_\alpha:\alpha\in I\}$ be Topological Spaces. Let *f:* $Y\to \Pi X_\alpha$ be a function. If f is αg-continuous, then $\pi_\alpha\bullet$ *f:* $Y\to X_\alpha$ is αg-continuous.

Converse of the above theorem is not true in general as shown by the following Example:

Example 2.1: Let X = {p, q, r, s}; τ_X = {ϕ, {p}, {q}, {s}, {p, q}, {p, s},{q, s}, {p, q, r}, {p, q, s}, X}, $Y_1 = Y_2$ = {a, b}; τ_{Y1} = {ϕ, {a}, Y_1}; τ_{Y2} = {ϕ, {a}, Y_2}; Y = $Y_1\times Y_2$ = {(a, a), (a, b), (b, a), (b, b)} and τ_Y = {ϕ, {(a, a)}, {(a, a), (a, b)}, {(a, a), (b, a)}, {(a, a), (a, b), (b, a)}, $Y_1\times Y_2$}.

Define *f:* X→ Y by f(p) = (a, a), f(q) = (b, b), f(r) = (a, b), f(s) = (b, a). It is easy to see that $\pi_1\bullet$f and $\pi_2\bullet$f are αg-continuous. However {(b, b)} is closed in Y but f^{-1}({(b, b)}) = {q} is not αg-closed in X. Therefore f is not αg-continuous.

Theorem 2.2: If Y is $\alpha T_{1/2}$ and $\{X_\alpha:\alpha\in I\}$ be Topological Spaces. Let *f:* $Y\to \Pi X_\alpha$ be a function, then f is αg-continuous iff $\pi_\alpha\bullet$ *f:* $Y\to X_\alpha$ is αg-continuous.

Corollary 2.3: (i) Let $f_\alpha: X_\alpha\to Y_\alpha$ be a function and let *f:* $\Pi X_\alpha\to \Pi Y_\alpha$ be defined by $f(x_\alpha)_{\alpha\in I} = (f_\alpha(x_\alpha))_{\alpha\in I}$. If f is αg-continuous then each f_α is αg-continuous.
(ii) For each α, let X_α be $\alpha T_{1/2}$ and let $f_\alpha: X_\alpha\to Y_\alpha$ be a function and let *f:* $\Pi X_\alpha\to \Pi Y_\alpha$ be defined by $f(x_\alpha)_{\alpha\in I} = (f_\alpha(x_\alpha))_{\alpha\in I}$, then f is αg-continuous iff each f_α is αg-continuous.

3. αg_i spaces i = 0, 1, 2:

Definition 3.1: X is said to be
(i) a αg_0 space if for each pair of distinct points x, y of X, there exists a αg-open set G containing either of the point x or y.
(ii) a αg_1[resp: αg_2] space if for each pair of distinct points x, y of X there exists [resp: disjoint] αg-open sets G and H containing x and y respectively.

Note 2:
(i) $rT_i\to T_i\to \alpha_i\to \alpha g_i$, i = 0, 1, 2. but the converse is not true in general.
(ii)X is αg_2 → X is αg_1 → X is αg_0.

Example 3.1: Let X = {a, b, c} and τ = {ϕ, X}, then X is αg_i but not rT_0 and T_0, i = 0, 1, 2.for i = 0, 1, 2.

Example 3.2: Let X = {a, b, c, d} and τ = {ϕ, {b}, {a, b}, {b, c}, {a, b, c}, X} then X is not αg_i for i = 0, 1, 2.

Remark 3.1: If X is $\alpha T_{1/2}$ then αT_i and αg_i are one and the same for i = 0, 1, 2.

Theorem 3.1: The following are true
(i) Every [resp: regular open] open subspace of αg_i space is αg_i for i = 0, 1, 2.
(ii) The product of αg_i spaces is again αg_i for i = 0, 1, 2.
(iii) αg-continuous image of T_i [resp: rT_i] spaces is αg_i for i = 0, 1, 2.
(iv) X is αg_0 iff ∀ x∈ X, ∃ U∈ αGO(X) containing x such that the subspace U is αg_0.
(v) X is αg_0 iff distinct points of X have disjoint αg-closures.
(vi) If X is αg_1 then distinct points of X have disjoint αg-closures.

Theorem 3.2: The following are equivalent:
(i) X is αg_1.
(ii) Each one point set is αg-closed.
(iii)Each subset of X is the intersection of all αg-open sets containing it.
(iv) For any x∈ X, the intersection of all αg-open sets containing the point is the set {x}.

Theorem 3.3: Suppose x is a αg-limit point of a subset of A of a αg_1 space X. Then every neighborhood of x contains infinitely many distinct points of A.

Theorem 3.4: The following are true
(i) X is αg_2 iff the intersection of all αg-closed, αg-neighborhoods of each point of the space is reduced to that point.

(ii) If to each point $x\in X$, there exist a αg-closed, αg-open subset of X containing x which is also a αg_2 subspace of X, then X is αg_2.
(iii) If X is αg_2 then the diagonal Δ in $X\times X$ is αg-closed.
(iv) In αg_2-space, αg-limits of sequences, if exists, are unique.
(v) In a αg_2 space, a point and disjoint αg-compact subspace can be separated by disjoint αg-open sets.
(vi) Every αg-compact subspace of a αg_2 space is αg-closed.

Corollary 3.1: The following are true
(i) In a T_1 [resp: rT_1; g_1] space, each singleton set is αg-closed.
(ii) If X is T_1 [resp: rT_1; g_1] then distinct points of X have disjoint αg-closures.
(iii) If X is T_2 [resp: rT_2; g_2] then the diagonal Δin $X\times X$ is αg-closed.
(iv) Show that in a T_2 [resp: rT_2; g_2] space, a point and disjoint compact[resp: nearly-compact; g-compact] subspace can be separated by disjoint αg-open sets
(v) Every compact [resp: nearly-compact; g-compact] subspace of a T_2 [resp: rT_2; g_2] space is αg-closed.

Theorem 3.5: The following are true
(i) If *f:* $X\to Y$ is injective, αg-irresolute and Y is αg_i then X is αg_i, i = 0, 1, 2.
(ii) Let X be T_1 and *f:* $X\to Y$ be αg-closed surjection. Then X is αg_1.
(iii) Every αg-irresolute map from a αg-compact space into a αg_2 space is αg-closed.
(iv) Any αg-irresolute bijection from a αg-compact space onto a αg_2 space is a αgc-homeomorphism.
(v) Any αg-continuous bijection from a αg-compact space onto a αg_2 space is a αg-homeomorphism.
(vi) If *f:* $X\to Y$ is injective, αg-continuous and Y is T_i then X is αg_i, i = 0, 1, 2.
(vii) If *f:* $X\to Y$ is injective, r-irresolute[r-continuous] and Y is rT_i then X is αg_i, i = 0, 1, 2.
(viii)The property of being a space is αg_0 is a αg-Topological property.
(ix) Let *f:* $X\to Y$ is a αgc-homeomorphism, then X is αg_i if Y is αg_i, i = 0, 1, 2.

Theorem 3.6: The following are equivalent:
(i) X is αg_2.
(ii) For each pair $x\neq y\in X$ $\exists$ a αg-open, αg-closed set V such that $x\in V$ and $y\notin V$, and
(iii) For each pair $x\neq y\in X$ $\exists$ *f:* $X\to [0, 1]$ such that f(x) = 0 and f(y) = 1 and f is αg-continuous.

Theorem 3.7: If *f:* $X\to Y$ is αg-irresolute and Y is αg_2 then
(i) the set $A = \{(x_1, x_2): f(x_1) = f(x_2)\}$ is αg-closed in $X\times X$.
(ii) G (f), graph of f, is αg-closed in $X\times Y$.

Theorem 3.8: If *f:* $X\to Y$ is αg-open and $A = \{(x_1, x_2): f(x_1) = f(x_2)\}$ is closed in $X\times X$. Then Y is αg_2.

Theorem 3.9: Let Y and $\{X_\alpha:\alpha\in I\}$ be Topological Spaces. If *f:* $Y\to \Pi X_\alpha$ be a αg-continuous function and Y is $\alpha T_{1/2}$, then ΠX_α and each X_α are αg_i, i = 0,1,2.

Theorem 3.10: Let X be an arbitrary space, R an equivalence relation in X and *p*: $X\to X/R$ the identification map. If $R\subset X\times X$ is αg-closed in $X\times X$ and *p* is αg-open map, then X/R is αg_2.

Theorem 3.11: The following four properties are equivalent:
(i) X is αg_2
(ii) Let $x\in X$. For each $y\neq x$, $\exists$ $U\in \alpha GO(X)$ such that $x\in U$ and $y\notin \alpha gcl(U)$
(iii) For each $x\in X$, $\cap\{\alpha gcl(U)/U\in \alpha GO(X)$ and $x\in U\} = \{x\}$.
(iv) The diagonal $\Delta = \{(x, x)/x\in X\}$ is αg-closed in $X\times X$.

Proof: (i) $\Rightarrow$ (ii) Let $x\in X$ and $y\neq x$. Then there are disjoint αg-open sets U and V such that $x\in U$ and $y\in V$. Clearly V^c is αg-closed, $\alpha gcl(U)\subset V^c$, $y\notin V^c$ and therefore $y\notin \alpha gcl(U)$.

(ii) $\Rightarrow$ (iii) If $y\neq x$, then $\exists$ $U\in \alpha GO(X)$ s.t. $x\in U$ and $y\notin \alpha gcl(U)$. So $y\notin\cap\{\alpha gcl(U)/U\in \alpha GO(X)$ and $x\in U\}$.

(iii) $\Rightarrow$ (iv) We prove Δ^c is αg-open. Let $(x, y)\notin\Delta$. Then $y\neq x$ and $\cap\{\alpha gcl(U)/U\in \alpha GO(X)$ and $x\in U\} = \{x\}$ there is some $U\in \alpha GO(X)$ with $x\in U$ and $y\notin \alpha gcl(U)$. Since $U\cap(\alpha gcl(U))^c = \phi$, $U\times(\alpha gcl(U))^c$ is a αg-open set such that $(x, y)\in U\times(\alpha gcl(U))^c\subset\Delta^c$.

(iv) $\Rightarrow$ (i) $y\neq x$, then $(x, y)\notin\Delta$ and thus there exist αg-open sets U and V such that $(x, y)\in U\times V$ and $(U\times V)\cap\Delta = \phi$.

Clearly, for the αg-open sets U and V we have; $x\in U$, $y\in V$ and $U\cap V = \phi$.

4. αg-R_i spaces; i = 0, 1:

Definition 4.1: Let $x \in X$. Then
(i) αg-kernel of x is defined and denoted by $Ker_{\{\alpha g\}}\{x\} = \cap\{U : U \in \alpha GO(X) \text{ and } x \in U\}$
(ii) $Ker_{\{\alpha g\}}F = \cap\{U: U \in \alpha GO(X) \text{ and } F \subset U\}$

Lemma 4.1: Let $A \subset X$, then $Ker_{\{\alpha g\}}\{A\} = \{x \in X: \alpha gcl\{x\} \cap A \neq \phi.\}$

Lemma 4.2: Let $x \in X$. Then $y \in Ker_{\{\alpha g\}}\{x\}$ iff $x \in \alpha gcl\{y\}$.

Proof: Let $y \notin Ker_{\{\alpha g\}}\{x\}$. Then $\exists\ V \in \alpha gO(X, x)$ such that $y \notin V$. Therefore we have $x \notin \alpha gcl\{y\}$. The proof of converse part can be done similarly.

Lemma 4.3: For any points $x \neq y \in X$, the following are equivalent:
(i) $Ker_{\{\alpha g\}}\{x\} \neq Ker_{\{\alpha g\}}\{y\}$;
(ii) $\alpha gcl\{x\} \neq \alpha gcl\{y\}$.

Proof:
(i) $\Rightarrow$ (ii): Let $Ker_{\{\alpha g\}}\{x\} \neq Ker_{\{\alpha g\}}\{y\}$, then $\exists\ z \in X$ such that $z \in Ker_{\{\alpha g\}}\{x\}$ and $z \notin Ker_{\{\alpha g\}}\{y\}$. From $z \in Ker_{\{\alpha g\}}\{x\}$ it follows that $\{x\} \cap \alpha gcl\{z\} \neq \phi \Rightarrow x \in \alpha gcl\{z\}$. By $z \notin Ker_{\{\alpha g\}}\{y\}$, we have $\{y\} \cap \alpha gcl\{z\} = \phi$. Since $x \in \alpha gcl\{z\}$, $\alpha gcl\{x\} \subset \alpha gcl\{z\}$ and $\{y\} \cap \alpha gcl\{x\} = \phi$. Therefore $\alpha gcl\{x\} \neq \alpha gcl\{y\}$.

Now $Ker_{\{\alpha g\}}\{x\} \neq Ker_{\{\alpha g\}}\{y\} \Rightarrow \alpha gcl\{x\} \neq \alpha gcl\{y\}$.

(ii) $\Rightarrow$ (i): If $\alpha gcl\{x\} \neq \alpha gcl\{y\}$. Then $\exists\ z \in X$ such that $z \in \alpha gcl\{x\}$ and $z \notin \alpha gcl\{y\}$. Then $\exists$ a αg-open set containing z and therefore containing x but not y, namely, $y \notin Ker_{\{\alpha g\}}\{x\}$. Hence $Ker_{\{\alpha g\}}\{x\} \neq Ker_{\{\alpha g\}}\{y\}$.

Definition 4.2: X is said to be
(i) αg-R_0 iff $\alpha gcl\{x\} \subseteq G$ whenever $x \in G \in \alpha GO(X)$.
(ii) weakly αg-R_0 iff $\cap\ \alpha gcl\{x\} = \phi$.
(iii) αg-R_1 iff for $x, y \in X \ni \alpha gcl\{x\} \neq \alpha gcl\{y\}\ \exists$ disjoint U; $V \in \alpha GO(X) \ni \alpha gcl\{x\} \subseteq U$ and $\alpha gcl\{y\} \subseteq V$.

Example 4.1: Let X = {a, b, c, d} and $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$, then X is weakly αgR_0 and not αgR_i, i = 0, 1.

Remark 4.1:
(i) r-$R_i \Rightarrow R_i \Rightarrow \alpha R_i \Rightarrow \alpha gR_i$, i = 0, 1.
(ii) Every weakly-R_0 space is weakly αgR_0.

Lemma 4.1: Every αgR_0 space is weakly αgR_0.

Converse of the above Theorem is not true in general by the following Examples.

Example 4.2: Let X = {a, b, c, d} and $\tau = \{\phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}$. Clearly, X is weakly αgR_0, since $\cap\ \alpha gcl\{x\} = \phi$. But it is not αgR_0, for $\{a\} \subset X$ is αg-open and $\alpha gcl\{a\} = \{a, b\} \not\subset \{a\}$.

Theorem 4.1: Every αg-regular space X is αg_2 and αg-R_0.

Proof: Let X be αg-regular and let $x \neq y \in X$. By Lemma 4.1, {x} is either αg-open or αg-closed. If {x} is αg-open, {x} is αg-open and hence αg-clopen. Thus {x} and X - {x} are separating αg-open sets. Similarly for {x} is αg-closed, {x} and X - {x} are separating αg-closed sets. Thus X is αg_2 and αg-R_0.

Theorem 4.2: X is αg-R_0 iff given $x \neq y \in X$; $\alpha gcl\{x\} \neq \alpha gcl\{y\}$.

Proof: Let X be αg-R_0 and let let $x, \neq y \in X$. Suppose U is a αg-open set containing x but not y, then $y \in \alpha gcl\{y\} \subset X - U$ and so $x \notin \alpha gcl\{y\}$. Hence $\alpha gcl\{x\} \neq \alpha gcl\{y\}$.

Conversely, let $x, \neq y \in X$ such that $\alpha gcl\{x\} \neq \alpha gcl\{y\} \Rightarrow \alpha gcl\{x\} \subset X - \alpha gcl\{y\} = U$(say) a αg-open set in X. This is true for every $\alpha gcl\{x\}$. Thus $\cap\ \alpha gcl\{x\} \subseteq U$ where $x \in \alpha gcl\{x\} \subseteq U \in \alpha GO(X)$, which in turn implies $\cap\ \alpha gcl\{x\} \subseteq U$ where $x \in U \in \alpha GO(X)$. Hence X is αgR_0.

Theorem 4.3: X is weakly αgR_0 iff $Ker_{\{\alpha g\}}\{x\} \neq X$ for any $x \in X$.

Proof: Let $x_0 \in X$ such that $ker_{\{\alpha g\}}\{x_0\} = X$. This means that x_0 is not contained in any proper αg-open subset of X. Thus x_0 belongs to the αg-closure of every singleton set. Hence $x_0 \in \cap \alpha gcl\{x\}$, a contradiction.

Conversely assume $Ker_{\{\alpha g\}}\{x\} \neq X$ for any $x \in X$. If there is an $x_0 \in X$ s.t. $x_0 \in \cap\{\alpha gcl\{x\}\}$, then every αg-open set containing x_0 must contain every point of X. Therefore, the unique αg-open set containing x_0 is X. Hence $Ker_{\{\alpha g\}}\{x_0\} = X$, which is a contradiction. Thus X is weakly αg-R_0.

Theorem 4.4: The following are equivalent:
(i) X is αg-R_0 space.
(ii) For each $x \in X$, $\alpha gcl\{x\} \subset Ker_{\{\alpha g\}}\{x\}$
(iii) For any αg-closed set F and a point $x \notin F$, $\exists\, U \in \alpha GO(X)$ such that $x \notin U$ and $F \subset U$.
(iv) Each αg-closed set F can be expressed as $F = \cap\{G: G$ is αg-open and $F \subset G\}$.
(v) Each αg-open set Gcan be expressed as $G = \cup\{A: A$ is αg-closed and $A \subset G\}$.
(vi) For each αg-closed set F, $x \notin F$ implies $\alpha g\text{-}cl\{x\} \cap F = \phi$.

Proof:
(i) ⇒ (ii) For any $x \in X$, we have $Ker_{\{\alpha g\}}\{x\} = \cap\{U: U \in \alpha GO(X)$ and $x \in U\}$. Since X is αg-R_0, each αg-open set containing x contains $\alpha gcl\{x\}$. Hence $\alpha gcl\{x\} \subset Ker_{\{\alpha g\}}\{x\}$.

(ii) ⇒ (iii) Let $x \notin F \in \alpha gc(X)$. Then for any $y \in F$; $\alpha gcl\{y\} \subset F$ and so $x \notin \alpha gcl\{y\} \Rightarrow y \notin \alpha gcl\{x\}$ that is $\exists\, U_y \in \alpha GO(X)$ such that $y \in U_y$ and $x \notin U_y$ $\forall\, y \in F$. Let $U = \cup\{U_y$: U_y is αg-open, $y \in U_y$ and $x \notin U_y\}$. Then U is αg-open such that $x \notin U$ and $F \subset U$.

(iii) ⇒ (iv) Let F be any αg-closed set and $N = \cap\{G: G$ is αg-open and $F \subset G\}$. Then $F \subset N \rightarrow (1)$.

Let $x \notin F$, then by (iii) $\exists\, G \in \alpha GO(X)$ such that $x \notin G$ and $F \subset G$.

Hence $x \notin N$ which implies $x \in N \Rightarrow x \in F$. Hence $N \subset F \rightarrow (2)$.

Therefore from (1) and (2), each αg-closed set $F = \cap\{G: G$ is αg-open and $F \subset G\}$

(iv) ⇒ (v) obvious.

(v) ⇒ (vi) Let $x \notin F \in \alpha gc(X)$. Then $X\text{-}F = G$ is a αg-open set containing x. Then by (v), Gcan be expressed as the union of αg-closed sets A contained in G, and so there is an $M \in \alpha gc(X)$ such that $x \in M \subset G$; and hence $\alpha gcl\{x\} \subset G$ which implies $\alpha gcl\{x\} \cap F = \phi$.

(vi) ⇒ (i) Let $x \in G \in \alpha GO(X)$. Then $x \notin (X\text{-}G)$, which is a αg-closed set. Therefore by (vi) $\alpha gcl\{x\} \cap (X\text{-}G) = \phi$, which implies that $\alpha gcl\{x\} \subseteq G$. Thus X is αgR_0 space.

Theorem 4.5: Let f: $X \rightarrow Y$ be a αg-closed one-one function. If X is weakly αg-R_0, then so is Y.

Theorem 4.6: If X is weakly αg-R_0, then for every space Y, $X \times Y$ is weakly αg-R_0.

Proof: $\cap\, \alpha gcl\{(x,y)\} \subseteq \cap\{\, \alpha gcl\{x\} \times \alpha gcl\{y\}\} = \cap\, [\alpha gcl\{x\}] \times [\alpha gcl\{y\}] \subseteq \phi \times Y = \phi$. Hence $X \times Y$ is αgR_0.

Corollary 4.1:
(i) If X and Y are weakly αgR_0, then $X \times Y$ is weakly αgR_0.
(ii) If X and Y are (weakly-)R_0, then $X \times Y$ is weakly αgR_0.
(iii)If X and Y are αgR_0, then $X \times Y$ is weakly αgR_0.
(iv) If X is αgR_0 and Y are weakly R_0, then $X \times Y$ is weakly αgR_0.

Theorem 4.7: X is αgR_0 iff for any $x, y \in X$, $\alpha gcl\{x\} \neq \alpha gcl\{y\} \Rightarrow \alpha gcl\{x\} \cap \alpha gcl\{y\} = \phi$.

Proof: Let X be αgR_0 and $x, y \in X$ such that $\alpha gcl\{x\} \neq \alpha gcl\{y\}$ .Then $\exists\, z \in \alpha gcl\{x\}$ such that $z \notin \alpha gcl\{y\}$ (or $z \in \alpha gcl\{y\}$) such that $z \notin \alpha gcl\{x\}$. There exists $V \in \alpha GO(X)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, $x \notin \alpha gcl\{y\}$. Thus $x \in [\, \alpha gcl\{y\}]^c \in \alpha GO(X)$, which implies $\alpha gcl\{x\} \subset [\, \alpha gcl\{y\}]^c$ and $\alpha gcl\{x\} \cap \alpha gcl\{y\} = \phi$. The proof for otherwise is similar.

Sufficiency: Let $x\in V\in \alpha GO(X)$. We show that $\alpha gcl\{x\}\subset V$. Let $y\notin V$, i.e., $y\in V^c$. Then $x \neq y$ and $x\notin \alpha gcl\{y\}$. Hence $\alpha gcl\{x\} \neq \alpha gcl\{y\}$. But $\alpha gcl\{x\}\cap \alpha gcl\{y\} = \phi$. Hence $y\notin \alpha gcl\{x\}$. Hence $\alpha gcl\{x\}\subset V$.

Theorem 4.8: X is αgR_0 iff for any points $x, y\in X$, $Ker_{\{\alpha g\}}\{x\} \neq Ker_{\{\alpha g\}}\{y\} \Rightarrow Ker_{\{\alpha g\}}\{x\}\cap Ker_{\{\alpha g\}}\{y\} = \phi$.

Proof: Let X be αgR_0. By Lemma 4.3 for any $x, y\in X$ if $Ker_{\{\alpha g\}}\{x\} \neq Ker_{\{\alpha g\}}\{y\}$ then $\alpha gcl\{x\} \neq \alpha gcl\{y\}$.Assume that $z\in Ker_{\{\alpha g\}}\{x\}\cap Ker_{\{\alpha g\}}\{y\}$. By $z\in Ker_{\{\alpha g\}}\{x\}$ and Lemma 4.2, it follows that $x\in \alpha gcl\{z\}$. Since $x\in \alpha gcl\{z\}$, $\alpha gcl\{x\} = \alpha gcl\{z\}$. Similarly, we have $\alpha gcl\{y\} = \alpha gcl\{z\} = \alpha gcl\{x\}$. This is a contradiction. Therefore, we have $Ker_{\{\alpha g\}}\{x\} \cap Ker_{\{\alpha g\}}\{y\} = \phi$.

Conversely, let $x, y\in X$, s.t. $\alpha gcl\{x\} \neq \alpha gcl\{y\}$, then by Lemma 4.3, $Ker_{\{\alpha g\}}\{x\} \neq Ker_{\{\alpha g\}}\{y\}$. Hence by hypothesis $Ker_{\{\alpha g\}}\{x\}\cap Ker_{\{\alpha g\}}\{y\} = \phi$ which implies $\alpha gcl\{x\}\cap \alpha gcl\{y\} = \phi$ Because $z\in \alpha gcl\{x\}$ implies that $x\in Ker_{\{\alpha g\}}\{z\}$ and therefore $Ker_{\{\alpha g\}}\{x\}\cap Ker_{\{\alpha g\}}\{z\} \neq \phi$ Therefore by Theorem 4.7 X is a αgR_0 space.

Theorem 4.9: The following are equivalent:
(i) X is a αg-R_0 space.
(ii) For any $A \neq \phi$ and $G\in \alpha GO(X)$ such that $A\cap G \neq \phi$ $\exists$ $F\in \alpha gc(X)$such that $A\cap F \neq \phi$ and $F\subset G$.

Proof:
(i) ⇒ (ii): Let $A \neq \phi$ and $G\in \alpha GO(X)$ such that $A\cap G \neq \phi$. There exists $x\in A\cap G$. Since $x\in G\in \alpha GO(X)$, $\alpha gcl\{x\}\subset G$. Set $F = \alpha gcl\{x\}$, then $F\in \alpha gc(X)$, $F\subset G$ and $A\cap F \neq \phi$

(ii) ⇒ (i): Let $G\in \alpha GO(X)$ and $x\in G$. By (2), $\alpha gcl\{x\}\subset G$. Hence X is αg-R_0.

Theorem 4.10: The following are equivalent:
(i) X is a αg-R_0 space;
(ii) $x\in \alpha gcl\{y\}$ iff $y\in \alpha gcl\{x\}$, for any points x and y in X.

Proof:
(i) ⇒ (ii): Assume X is αgR_0. Let $x\in \alpha gcl\{y\}$ and D be any αg-open set such that $y\in D$. Now by hypothesis, $x\in D$. Therefore, every αg-open set which contain y contains x. Hence $y\in \alpha gcl\{x\}$.

(ii) ⇒ (i): Let U be a αg-open set and $x\in U$. If $y\notin U$, then $x\notin \alpha gcl\{y\}$ and hence $y\notin \alpha gcl\{x\}$. This implies that $\alpha gcl\{x\}\subset U$. Hence X is αgR_0.

Theorem 4.11: The following are equivalent:
(i) X is a αgR_0 space;
(ii) If F is αg-closed, then $F = Ker_{\{\alpha g\}}(F)$;
(iii)If F is αg-closed and $x\in F$, then $Ker_{\{\alpha g\}}\{x\}\subseteq F$;
(iv) If $x\in X$, then $Ker_{\{\alpha g\}}\{x\}\subset \alpha gcl\{x\}$.

Proof:
(i) ⇒ (ii): Let $x\notin F\in \alpha gc(X) \Rightarrow (X\text{-}F)\in \alpha gO(X, x)$. For X is αgR_0, $\alpha gcl(\{x\})\subset(X\text{-}F)$. Thus $\alpha gcl(\{x\})\cap F = \phi$ and $x\notin Ker_{\{\alpha g\}}(F)$. Hence $Ker_{\{\alpha g\}}(F) = F$.

(ii) ⇒ (iii): $A\subset B \Rightarrow Ker_{\{\alpha g\}}(A)\subset Ker_{\{\alpha g\}}(B)$. Therefore, by (2) $Ker_{\{\alpha g\}}\{x\}\subset Ker_{\{\alpha g\}}(F) = F$.

(iii) ⇒ (iv): Since $x\in \alpha gcl\{x\}$ and $\alpha gcl\{x\}$ is αg-closed, by (3) $Ker_{\{\alpha g\}}\{x\}\subset \alpha gcl\{x\}$.

(iv) ⇒ (i): Let $x\in \alpha gcl\{y\}$. Then by Lemma 4.2 $y\in Ker_{\{\alpha g\}}\{x\}$. Since $x\in \alpha gcl\{x\}$ and $\alpha gcl\{x\}$ is αg-closed, by (iv) we obtain $y\in Ker_{\{\alpha g\}}\{x\}\subseteq \alpha gcl\{x\}$. Therefore $x\in \alpha gcl\{y\}$ implies $y\in \alpha gcl\{x\}$. The converse is obvious and X is αgR_0.

Corollary 4.2: The following are equivalent:
(i) X is αgR_0.
(ii) $\alpha gcl\{x\} = Ker_{\{\alpha g\}}\{x\}\forall x\in X$.

Proof: Follows from Theorems 4.4 and 4.11.

Recall that a filterbase F is called αg-convergent to a point x in X, if for any αg-open set U of X containing x, there exists $B\in F$ such that $B\subset U$.

Lemma 4.4: Let x and y be any two points in X such that every net in X αg-converging to y αg-converges to x. Then $x\in \alpha gcl\{y\}$.

Theorem 4.12: The following are equivalent:
(i) X is a αgR_0 space;
(ii) If $x, y\in X$, then $y\in \alpha gcl\{x\}$ iff every net in X αg-converging to y αg-converges to x.

Proof:
(i) ⇒ (ii): Let $x, y\in X \ni y\in \alpha gcl\{x\}$. If $\{x_\alpha\}_{\alpha\in I}$ is a net in $X \ni \{x_\alpha\}_{\alpha\in I}$ αg-converges to y. Since $y\in \alpha gcl\{x\}$, by Thm. 4.7 we have $\alpha gcl\{x\} = \alpha gcl\{y\}$. Therefore $x\in \alpha gcl\{y\}$. This means that $\{x_\alpha\}_{\alpha\in I}$ αg-converges to x.

Conversely, let $x, y\in X \ni$ every net in X αg-converging to y αg-converges to x. Then $x\in \alpha g\text{-}cl\{y\}$[by 4.4]. By Thm. 4.7, we have $\alpha gcl\{x\} = \alpha gcl\{y\}$. Therefore $y\in \alpha gcl\{x\}$.

(ii) ⇒ (i): Let $x, y\in X \ni \alpha gcl\{x\}\cap \alpha gcl\{y\} \neq \phi$. Let $z\in \alpha gcl\{x\}\cap \alpha gcl\{y\}$. So $\exists$ a net $\{x_\alpha\}_{\alpha\in I}$ in $\alpha gcl\{x\} \ni \{x_\alpha\}_{\alpha\in I}$ αg-converges to z. Since $z\in \alpha gcl\{y\}$, then $\{x_\alpha\}_{\alpha\in I}$ αg-converges to y. It follows that $y\in \alpha gcl\{x\}$. Similarly we obtain $x\in\alpha gcl\{y\}$. Therefore $\alpha gcl\{x\} = \alpha gcl\{y\}$. Hence X is αgR_0.

Theorem 4.13:
(i) Every subspace of αgR_1 space is again αgR_1.
(ii) Product of any two αgR_1 spaces is again αgR_1.
(iii) X is αgR_1 iff given $x \neq y\in X$, $\alpha gcl\{x\} \neq \alpha gcl\{y\}$.
(iv) Every αg_2 space is αgR_1.

The converse of 4.13(iv) is not true. However, we have the following result.

Theorem 4.14: Every αg_1 and αgR_1 space is αg_2.

Proof: Let $x \neq y\in X$. Since X is αg_1; {x} and {y} are αg-closed sets s.t. $\alpha gcl\{x\} \neq \alpha gcl\{y\}$. Since X is αgR_1, there exists disjoint αg-open sets U and V s.t. $x\in U$; $y\in V$. Hence X is αg_2.

Corollary 4.3: X is αg_2 iff it is αgR_1 and αg_1.

Theorem 4.15: The following are equivalent
(i) X is $\alpha g\text{-}R_1$.
(ii) $\cap\ \alpha gcl\{x\} = \{x\}$.
(iii) For any $x\in X$, intersection of all αg-neighborhoods of x is {x}.

Proof:
(i) ⇒ (ii) Let $y \neq x\in X$ such that $y\in \alpha gcl\{x\}$. Since X is αgR_1, $\exists\ U\in \alpha GO(X)$ such that $y\in U$, $x\notin U$ or $x\in U$, $y\notin U$. In either case $y\notin \alpha gcl\{x\}$. Hence $\cap\ \alpha gcl\{x\} = \{x\}$.

(ii) ⇒ (iii) If $y \neq x\in X$, then $x\notin\cap\ \alpha gcl\{y\}$, so there is a αg-open set containing x but not y. Therefore y does not belong to the intersection of all αg-neighborhoods of x. Hence intersection of all αg-neighborhoods of x is {x}.

(iii) ⇒ (i) Let $x \neq y\in X$. by hypothesis, y does not belong to the intersection of all αg-neighborhoods of x and x does not belong to the intersection of all αg-neighborhoods of y, which implies $\alpha gcl\{x\} \neq \alpha gcl\{y\}$. Hence X is $\alpha g\text{-}R_1$.

Theorem 4.16: The following are equivalent:
(i) X is $\alpha g\text{-}R_1$.
(ii) For each pair $x, y\in X$ with $\alpha gcl\{x\} \neq \alpha gcl\{y\}$, $\exists$ a αg-open, αg-closed set V s.t. $x\in V$ and $y\notin V$, and
(iii)For each pair $x, y\in X$ with $\alpha gcl\{x\} \neq \alpha gcl\{y\}$, $\exists\ f: X\to[0, 1]$ s.t. $f(x) = 0$ and $f(y) = 1$ and f is αg-continuous.

Theorem 4.17:
(i) If X is $\alpha g\text{-}R_1$, then X is $\alpha g\text{-}R_0$.
(ii) X is $\alpha g\text{-}R_1$ iff for $x, y\in X$, $Ker_{\{\alpha g\}}\{x\} \neq Ker_{\{\alpha g\}}\{y\}$, $\exists$ disjoint $U; V\in \alpha GO(X)$ such that $\alpha gcl\{x\}\subset U$ and $\alpha gcl\{y\}\subset V$.

5. $\alpha g\text{-}C_i$ and $\alpha g\text{-}D_i$ spaces, i = 0, 1, 2:

Definition 5.1: X is said to be a

(i) αg-C_0 space if for each pair of distinct points x, y of X there exists a αg-open set G whose closure contains either x or y.
(ii) αg-C_1[resp: αg-C_2] space if for each pair of distinct points x, y of X there exists [resp: disjoint] αg-open sets G and H whose closures containing x and y respectively.

Note: αg-$C_2 \Rightarrow$ αg-$C_1 \Rightarrow \alpha$g-C_0. Converse need not be true in general:

Example 5.1: (i) Let X = {a, b, c} and $\tau = \{\phi, X\}$, then X is αgC_i for i = 0, 1, 2.
(ii) Let X = {a, b, c, d} and $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ then X is not αgC_i for i = 0, 1, 2.

Theorem 5.1: We have the following properties
(i) Every subspace of αg-C_i space is αg-C_i.
(ii) Every αg_i spaces is αg-C_i.
(iii) Product of αg-C_i spaces are αg-C_i.
(iv) Let X be any αg-C_i space and $A \subset X$ then A is αg-C_i iff (A, $\tau_{/A}$) is αg-C_i.
(v) If X is αg-C_1 then each singleton set is αg-closed.
(vi) In an αg-C_1 space disjoint points of X has disjoint αg- closures.

Definition 5.2: $A \subset X$ is called a αg-Difference(Shortly αgD-set) set if there are two U, V$\in \alpha GO(X)$ such that $U \neq X$ and A = U-V.

Clearly every αg-open set U different from X is a αgD-set if A = U and V = ϕ.

Definition 5.3: X is said to be a
(i) αg-D_0 if for any pair of distinct points x and y of X there exist a αgD-set in X containing x but not y or a αgD-set in X containing y but not x.
(ii) αg-D_1 [resp: αg-D_2] if for any pair of distinct points x and y of X there exists [resp: disjoint] αgD-sets G and H in X containing x and y respectively.

Remark 5.2: (i) If X is rT_i, then it is αg_i, i = 0, 1, 2 and converse is false.
(ii) If X is αg_i, then it is $\alpha g_{\{i-1\}}$, i = 1, 2.
(iii) If X is αg_i, then it is αg-D_i , i = 0, 1, 2.
(iv) If X is αg-D_i, then it is αg-$D_{\{i-1\}}$, i = 1, 2.

Theorem 5.2: The following are true:
(i) X is αg-D_0 iff it is αg_0.
(ii) X is αg-D_1 iff it is αg-$D_{2.}$

Corollary 5.1: If X is αg-D_1, then it is αg_0.

Proof: Remark 5.1(iv) and Theorem 5.1(vi)

Definition 5.4: A point $x\in X$ which has X as the unique αg-neighborhood is called αg.c.c point.

Theorem 5.3: For an αg_0 space X the following are equivalent:
(i) X is αg-D_1;
(ii) X has no αg.c.c point.

Proof: (i) $\Rightarrow$ (ii) Since X is αg-D_1, then each point x of X is contained in a αgD-set O = U - V and thus in U. By Definition $U \neq X$. This implies that x is not a αg.c.c point.

(ii) $\Rightarrow$ (i) If X is αg_0, then for each $x \neq y\in X$, at least one of them, x (say) has a αg-neighborhood U containing x and not y. Thus U which is different from X is a αgD-set. If X has no αg.c.c point, then y is not a αg.c.c point. This means that there exists a αg-neighborhood V of y such that $V \neq X$. Thus $y\in$ V-U but not x and V-U is a αgD-set. Hence X is αg-$D_{1.}$

Definition 5.5: X is αg-symmetric if for x and y in X, $x \in \alpha gcl\{y\}$ implies $y \in \alpha gcl\{x\}$.

Theorem 5.4: X is αg-symmetric iff {x} is αg-closed for each $x\in X$.

Proof: Assume that $x\in \alpha gcl\{y\}$ but $y\notin \alpha gcl\{x\}$. Then $[\alpha gcl\{x\}]^c$ contains y. This implies that $\alpha gcl\{y\}\subset [\alpha gcl\{x\}]^c$.

Now $[\alpha gcl\{x\}]^c$ contains x which is a contradiction.

Conversely, suppose $\{x\}\subset E\in\alpha GO(X)$ but $\alpha gcl\{x\} \not\subset E$. Then $\alpha gcl\{x\}$ and E^c are not disjoint. Let y belongs to their intersection. Now we have $x\in \alpha gcl\{y\}\subset E^c$ and $x\notin E$. But this is a contradiction.

Corollary 5.2: If X is a αg_1, then it is αg-symmetric.

Proof: Follows from Theorem 2.2(ii) and Theorem 5.4

Corollary 5.3: The following are equivalent:
(i) X is αg-symmetric and αg_0
(ii) X is αg_1.

Proof: By Corollary 5.2 and Remark 5.1 it suffices to prove only (i) $\Rightarrow$ (ii). Let $x \neq y$ and by αg_0, we may assume that $x\in G_1\subset\{y\}^c$ for some $G_1\in\alpha GO(X)$.Then $x\notin\alpha gcl\{y\}$ and hence $y\notin\alpha gcl\{x\}$. There exists a $G_2\in\alpha GO(X)$ such that $y\in G_2 \subset\{x\}^c$ and X is a αg_1 space.

Theorem 5.5: For a αg-symmetric space X the following are equivalent:
(i) X is αg_0; (ii) X is αg-D_1; (iii) X is αg_1.

Proof: (i) $\Rightarrow$ (iii) Corollary 5.4 and (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) Remark 5.1.

Theorem 5.6: If *f:* $X\rightarrow Y$ is αg-irresolute surjection and E is a αgD-set in Y, then $f^{-1}(E)$ is a αgD-set in X.

Theorem 5.7: If Y is αg-D_1 and *f:* $X \rightarrow Y$ is αg-irresolute and bijective, then X is αg-D_1.

Theorem 5.8: X is αg-D_1 iff for each $x \neq y$ in X there exist a αg-irresolute surjective function *f:* $X\rightarrow Y$, where Y is a αg-D_1 space such that $f(x)$ and $f(y)$ are distinct.

Corollary 5.4: Let $\{X_\alpha/\alpha\in I\}$ be any family of spaces. If X_α is αg-D_1 for each $\alpha\in I$, then ΠX_α is αg-D_1.

References

[1] S. P. Arya and T. Nour, Characterizations of s-normal spaces, I.J.P.A.M.,21(8)(1990),717-719.

[2] S. Balasubramanian, g-separation axioms, Scientia Magna, **6(4)** (2010)1-14.

[3] K. Balachandran, P. Sundaram and H. Maki, On generalized continuous maps in Topological Spaces, Mem. Fac. Sci. Kochi. Univ(Math)12(1991)05-13.

[4] Chawalit Boonpok-Generalized continuous functions from any topological space into product, Naresuan University journal (2003)11(2)93-98.

[5] Chawalit Boonpok, Preservation Theorems concering g-Hausdorf and rg-Hausdorff spaces, KKU. Sci.J.31 (3) (2003)138-140.

[6] R. Devi, K. Balachandran and H. Maki, semi-Generalized Homeomorphisms and Generalized semi-Homeomorphismin Topological Spaces, IJPAM, 26(3)(1995)271-284.

[7] W. Dunham, $T_{1/2}$ Spaces, Kyungpook Math. J. 17 (1977), 161-169.

[8] Norman Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo, 19 (2) (1970), 89-96**.**

[9] T. Noiri and V. Popa, On G-regular spaces and some functions, Mem. Fac. Sci. Kochi. Univ(Math)20(1999)67-74.

[10] N. Palaniappan and K. Chandrasekhara rao, Regular Generalized closed sets, Kyungpook M.J. Vol.33 (2) (1993)211-219.

****

