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On pg-Separation Axioms

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ABSTRACT

In this paper by using pg-open sets we define almost pg-normality and mild pg-normality also we continue the study of further properties of pg-normality. We show that these three axioms are regular open hereditary. We also define the class of almost pg-irresolute mappings and show that pg-normality is invariant under almost pg-irresolute M-pg-open continuous surjection.

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***Key words and Phrases**: pg-open, preopen, almost normal, midly normal, M-pg-closed, M-pg-open, pg-T_0-limit point.*

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1. INTRODUCTION

In 1967, A. Wilansky has introduced the concept of US spaces. In 1968, C.E. Aull studied some separation axioms between the T_1 and T_2 spaces, namely, S_1 and S_2. Next, in 1982, S.P. Arya et al have introduced and studied the concept of semi-US spaces and also they made study of s-convergence, sequentially semi-closed sets, sequentially s-compact notions. G.B. Navlagi studied P-Normal Almost-P-Normal, Mildly-P-Normal and Pre-US spaces. Recently S. Balasubramanian and P. Aruna Swathi Vyjayanthi studied *v*-Normal Almost- *v*-Normal, Mildly-*v*-Normal and *v*-US spaces. Inspired with these we introduce pg-Normal Almost- pg-Normal, Mildly- pg-Normal, pg-US, pg-S_1 and pg-S_2. Also we examine pg-convergence, sequentially pg-compact, sequentially pg-continuous maps, and sequentially sub pg-continuous maps in the context of these new concepts. All notions and symbols which are not defined in this paper may be found in the appropriate references. Throughout the paper X and Y denote Topological spaces on which no separation axioms are assumed explicitly stated.

2. PRELIMINARIES

Definition 2.1: $A\subset X$ is called
(i) g-closed if cl $A\subseteq U$ whenever $A\subseteq U$ and U is open in X.
(ii) pg-closed if $pcl(A) \subseteq U$ whenever $A\subseteq U$ and U is preopen in X.

Definition 2.2: A space X is said to be
(i) T_1 (T_2) if for any $x \neq y$ in X, there exist (disjoint) open sets U; V in X such that $x\in U$ and $y\in V$.
(ii) weakly Hausdorff if each point of X is the intersection of regular closed sets of X.
(iii) normal [resp: mildly normal] if for any pair of disjoint [resp: regular-closed]closed sets F_1 and F_2 , there exist disjoint open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.
(iv) almost normal if for each closed set A and each regular closed set B such that $A\cap B = \phi$, there exist disjoint open sets U and V such that $A\subset U$ and $B\subset V$.
(v) weakly regular if for each pair consisting of a regular closed set A and a point x such that $A \cap \{x\} = \phi$, there exist disjoint open sets U and V such that $x \in U$ and $A\subset V$.
(vi) A subset A of a space X is S-closed relative to X if every cover of A by semiopen sets of X has a finite subfamily whose closures cover A.

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(vii) R_0 if for any point x and a closed set F with $x \notin F$ in X, there exists a open set G containing F but not x.
(viii) R_1 iff for x, y $\in$ X with cl{x} $\neq$ cl{y}, there exist disjoint open sets U and V such that cl{x}$\subset$ U, cl{y}$\subset$V.
(ix) US-space if every convergent sequence has exactly one limit point to which it converges.
(x) pre-US space if every pre-convergent sequence has exactly one limit point to which it converges.
(xi) pre-S_1 if it is pre-US and every sequence $<x_n>$ pre-converges with subsequence of $<x_n>$ pre-side points.
(xii) pre-S_2 if it is pre-US and every sequence $<x_n>$ in X pre-converges which has no pre-side point.
(xiii) is weakly countable compact if every infinite subset of X has a limit point in X.
(xiv) Baire space if for any countable collection of closed sets with empty interior in X, their union also has empty interior in X.

Definition 2.3: Let $A\subset X$. Then a point x is said to be a
(i) limit point of A if each open set containing x contains some point y of A such that x $\neq$ y.
(ii) T_0–limit point of A if each open set containing x contains some point y of A such that cl{x} $\neq$ cl{y}, or equivalently, such that they are topologically distinct.
(iii) *pre*-T_0–limit point of A if each open set containing x contains some point y of A such that *p*cl{x} $\neq$ *p*cl{y}, or equivalently, such that they are topologically distinct.

Note 1: Recall that two points are topologically distinguishable or distinct if there exists an open set containing one of the points but not the other; equivalently if they have disjoint closures. In fact, the T_0–axiom is precisely to ensure that any two distinct points are topologically distinct.

Example 1: Let X = {a, b, c, d} and τ = {{a}, {b, c}, {a, b, c}, X,ϕ}. Then b and c are the limit points but not the T_0–limit points of the set {b, c}. Further d is a T_0–limit point of {b, c}.

Example 2: Let X = (0, 1) and τ = {ϕ, X, and U_n = (0, 1–1/n), n = 2, 3, 4 . . .}. Then every point of X is a limit point of X. Every point of X~U_2 is a T_0–limit point of X, but no point of U_2 is a T_0–limit point of X.

Definition 2.4: A set A together with all its T_0–limit points will be denoted by T_0–cl$A$.

Note 2: i. Every T_0–limit point of a set A is a limit point of the set but the converse is not true in general.
ii. In T_0–space both are same.

Note 3: R_0–axiom is weaker than T_1–axiom. It is independent of the T_0–axiom. However $T_1 = R_0+T_0$

Note 4: Every countable compact space is weakly countable compact but converse is not true in general. However, a T_1–space is weakly countable compact iff it is countable compact.

3. pg-**T_0 LIMIT POINT:**

Definition 3.01: In X, a point x is said to be a pg-T_0–limit point of A if each pg-open set containing x contains some point y of A such that *pgcl*{x} $\neq$ *pgcl*{y}, or equivalently; such that they are topologically distinct with respect to pg-open sets.

Note 5: *regular open set* $\Rightarrow$ *open set* $\Rightarrow$ pre-*open set* $\Rightarrow$ pg-*open set we have*
r-T_0–limit point $\Rightarrow$ *T_0–limit point* $\Rightarrow$ pre-*T_0–limit point* $\Rightarrow$ pg-*T_0–limit point*

Example 3: Let X = {a, b, c, d} and τ = {ϕ, {b, d}, {a, b, d}, {b, c, d}, X}. For A = {a, b, d}, a is *pg-T_0*–limit point.

Definition 3.02: A set A together with all its pg-T_0–limit points is denoted by T_0-*pgcl*(A)

Lemma 3.01: *If x is a* pg-*T_0–limit point of a set A then x is* pg-*limit point of A.*

Lemma 3.02:
(i) If X is pg-*T_0–space then every* pg-*T_0–limit point and every* pg-*limit point are equivalent.*
(ii) If X is r-T_0–space then every pg-*T_0–limit point and every* pg-*limit point are equivalent.*

Theorem 3.03: *For x $\neq$ y $\in$X,*
(i) x is a pg-*T_0–limit point of {y} iff x$\notin$pgcl{y} and y$\in$pgcl{x}.*
(ii) x is not a pg-*T_0–limit point of {y} iff either x$\in$pgcl{y}or pgcl{x} = pgcl{y}.*
(iii) x is not a pg-*T_0–limit point of {y} iff either x$\in$pgcl{y}or y$\in$pgcl{x}.*

Corollary 3.04:

(i) *If x is a* pg-T_0*–limit point of {y}, then y cannot be a* pg-*limit point of {x}.*
(ii) *If pgcl{x} = pgcl{y}, then neither x is a* pg-T_0*–limit point of {y} nor y is a* pg-T_0*–limit point of {x}.*

(iii) *If a singleton set A has no* pg-T_0*–limit point in X, then pgclA = pgcl{x} for all* $x \in pgcl\{A\}$.

Lemma 3.05: *In X, if x is a* pg-*limit point of a set A, then in each of the following cases x becomes* pg-T_0*–limit point of A* ($\{x\} \neq A$).

(i) $pgcl\{x\} \neq pgcl\{y\}$ *for* $y \in A$, $x \neq y$.
(ii) $pgcl\{x\} = \{x\}$
(iii) *X is a* pg-T_0*–space.*
(iv) $A \sim \{x\}$ *is* pg-*open*

4. pg-$\mathbf{T_0}$ **AND** pg-$\mathbf{R_i}$ **AXIOMS, i = 0, 1:**

In view of Lemma 3.6(iii), *pg*-T_0–axiom implies the equivalence of the concept of limit point of a set with that of *pg*-T_0–limit point of the set. But for the converse, if $x \in$ *pg*cl{y} then *pg*cl{x} ≠ *pg*cl{y} in general, but if x is a *pg*-T_0–limit point of {y}, then *pg*cl{x} = *pg*cl{y}

Lemma 4.01: *In a space X, a limit point x of {y} is a pg-*T_0*–limit point of {y} iff* $pgcl\{x\} \neq pgcl\{y\}$.

This lemma leads to characterize the equivalence of *pg*-T_0–limit point and *pg*-limit point of a set as the *pg*-T_0–axiom.

Theorem 4.02: *The following conditions are equivalent:*

(i) *X is a pg-*T_0 *space*
(ii) *Every pg-limit point of a set A is a pg-*T_0*–limit point of A*
(iii) *Every r-limit point of a singleton set {x} is a pg-*T_0*–limit point of {x}*
(iv) *For any x, y in X,* $x \neq y$ *if* $x \in pgcl\{y\}$*, then x is a pg-*T_0*–limit point of {y}*

Note 6: In a *pg*-T_0–space X if every point of X is a r-limit point of X, then every point of X is *pg*-T_0–limit point of X. But a space X in which each point is a *pg*-T_0–limit point of X is not necessarily a *pg*-T_0–space

Theorem 4.03: *The following conditions are equivalent:*

(i) *X is a pg-*R_0 *space*
(ii) *For any x, y in X, if* $x \in pgcl\{y\}$*, then x is not a pg-*T_0*–limit point of {y}*
(iii) *A point pg-closure set has no pg-*T_0*–limit point in X*
(iv) *A singleton set has no pg-*T_0*–limit point in X.*

Theorem 4.04: *In a pg-*R_0 *space X, a point x is pg-*T_0*–limit point of A iff every pg-open set containing x contains infinitely many points of A with each of which x is topologically distinct*

Theorem 4.05: *X is pg-*R_0 *space iff a set A of the form* $A = \cup pgcl\{x_{i\ i=1\ to\ n}\}$ *a finite union of point closure sets has no pg-*T_0*–limit point.*

If *pg*-R_0 space is replaced by rR_0 space in the above theorem, we have the following corollaries:

Corollary 4.06: *The following conditions are equivalent:*

(i) *X is a r-*R_0 *space*
(ii) *For any x, y in X, if* $x \in pgcl\{y\}$*, then x is not a pg-*T_0*–limit point of {y}*
(iii) *A point pg-closure set has no pg-*T_0*–limit point in X*
(iv) *A singleton set has no pg-*T_0*–limit point in X.*

Corollary 4.07: *In an r*R_0*–space X,*

(i) *If a point x is r*T_0*–limit point of a set then every pg-open set containing x contains infinitely many points of A with each of which x is topologically distinct.*
(ii) *If a point x is pg-*T_0*–limit point of a set then every pg-open set containing x contains infinitely many points of A with each of which x is topologically distinct.*
(iii) *If* $A = \cup pgcl\{x_{i,\ i=1\ to\ n}\}$ *a finite union of point closure sets has no pg-*T_0*–limit point.*
(iv) *If* $X = \cup pgcl\{x_{i,\ i=1\ to\ n}\}$ *then X has no pg-*T_0*–limit point.*

Various characteristic properties of *pg*-T_0–limit points studied so far is enlisted in the following theorem.

Theorem 4.08: *In a pg-R_0–space, we have the following:*

(i) A singleton set has no pg-T_0–limit point in X.
(ii) A finite set has no pg-T_0–limit point in X.
(iii) A point pg-closure has no set pg-T_0–limit point in X
(iv) A finite union point pg-closure sets have no set pg-T_0–limit point in X.
(v) For $x, y \in X$, $x \in T_0$– $pgcl\{y\}$ iff $x = y$.
(vi) For any $x, y \in X$, $x \neq y$ iff neither x is pg-T_0–limit point of $\{y\}$nor y is pg-T_0–limit point of $\{x\}$
(vii) For any $x, y \in X$, $x \neq y$ iff T_0– $pgcl\{x\} \cap T_0$– $pgcl\{y\} = \phi$.
(viii) Any point $x \in X$ is a pg-T_0–limit point of a set A in X iff every pg-open set containing x contains infinitely many points of A with each which x is topologically distinct.

Theorem 4.09: *X is pg-R_1 iff for any pg-open set U in X and points x, y such that $x \in X\sim U$, $y \in U$, there exists a pg-open set V in X such that $y \in V \subset U$, $x \notin V$.*

Lemma 4.10: *In pg-R_1 space X, if x is a pg-T_0–limit point of X, then for any non empty pg-open set U, there exists a non empty pg-open set V such that $V \subset U$, $x \notin pgcl(V)$.*

Lemma 4.11: *In a pg- regular space X, if x is a pg-T_0–limit point of X, then for any non empty pg-open set U, there exists a non empty pg-open set V such that $pgcl(V) \subset U$, $x \notin pgcl(V)$.*

Corollary 4.12: *In a regular space X,*

(i) If x is a pg-T_0–limit point of X, then for any non empty pg-open set U, there exists a non empty pg-open set V such that $pgcl(V) \subset U$, $x \notin pgcl(V)$.
(ii) If x is a T_0–limit point of X, then for any non empty pg-open set U, there exists a non empty pg-open set V such that $pgcl(V) \subset U$, $x \notin pgcl(V)$.

Theorem 4.13: *If X is a pg-compact pg-R_1-space, then X is a Baire Space.*

Proof: Let $\{A_n\}$ be a countable collection of *pg*-closed sets of X, each A_n having empty interior in X. Take A_1, since A_1 has empty interior, A_1 does not contain any *pg*-open set say U_0. Therefore we can choose a point $y \in U_0$ such that $y \notin A_1$.

For X is *pg*-regular, and $y \in (X\sim A_1) \cap U_0$, a *pg*-open set, we can find a *pg*-open set U_1 in X such that $y \in U_1$, $pgcl(U_1) \subset (X\sim A_1) \cap U_0$. Hence U_1 is a non empty *pg*-open set in X such that $pgcl(U_1) \subset U_0$ and $pgcl(U_1) \cap A_1 = \phi$. Continuing this process, in general, for given non empty *pg*-open set U_{n-1}, we can choose a point of U_{n-1} which is not in the *pg*-closed set A_n and a *pg*-open set U_n containing this point such that $pgcl(U_n) \subset U_{n-1}$ and $pgcl(U_n) \cap A_n = \phi$. Thus we get a sequence of nested non empty *pg*-closed sets which satisfies the finite intersection property. Therefore $\cap\, pgcl(U_n) \neq \phi$.

Then some $x \in \cap\, pgcl(U_n)$ which in turn implies that $x \in U_{n-1}$ as $pgcl(U_n) \subset U_{n-1}$ and $x \notin A_n$ for each n.

Corollary 4.14: *If X is a compact pg-R_1-space, then X is a Baire Space.*

Corollary 4.15: *Let X be a pg-compact pg-R_1-space. If $\{A_n\}$ is a countable collection of pg-closed sets in X, each A_n having non-empty pg-interior in X, then there is a point of X which is not in any of the A_n.*

Corollary 4.16: *Let X be a pg-compact R_1-space. If $\{A_n\}$ is a countable collection of pg-closed sets in X, each A_n having non-empty pg- interior in X, then there is a point of X which is not in any of the A_n.*

Theorem 4.17: *Let X be a non empty compact pg-R_1-space. If every point of X is a pg-T_0–limit point of X then X is uncountable.*

Proof: Since X is non empty and every point is a *pg*-T_0-limit point of X, X must be infinite. If X is countable, we construct a sequence of *pg*- open sets $\{V_n\}$ in X as follows:

Let $X = V_1$, then for x_1 is a *pg*-T_0-limit point of X, we can choose a non empty *pg*-open set V_2 in X such that $V_2 \subset V_1$ and $x_1 \notin pgclV_2$. Next for x_2 and non empty *pg*-open set V_2, we can choose a non empty *pg*-open set V_3 in X such that $V_3 \subset V_2$ and $x_2 \notin pgclV_3$. Continuing this process for each x_n and a non empty *pg*-open set V_n, we can choose a non empty *pg*-open set V_{n+1} in X such that $V_{n+1} \subset V_n$ and $x_n \notin pgclV_{n+1}$.

Now consider the nested sequence of *pg*-closed sets $pgclV_1 \supset pgclV_2 \supset pgclV_3 \supset \ldots\ldots\ldots \supset pgclV_n \supset \ldots$ Since X is *pg*-compact and $\{pgclV_n\}$ the sequence of *pg*-closed sets satisfies finite intersection property. By Cantors intersection theorem, there exists an x in X such that $x \in pgclV_n$. Further $x \in X$ and $x \in V_1$, which is not equal to any of the points of X. Hence X is uncountable.

Corollary 4.18: *Let X be a non empty pg-compact pg-R_1-space. If every point of X is a pg-T_0–limit point of X then X is uncountable*

5. *pg*–T_0-IDENTIFICATION SPACES AND *pg*–SEPARATION AXIOMS

Definition 5.01: Let (X, τ) be a topological space and let $\Re$ be the equivalence relation on X defined by $x\Re y$ iff *pgcl*{x} = *pgcl*{y}

Problem 5.02: show that $x\Re y$ iff *pgcl*{x} = *pgcl*{y} is an equivalence relation

Definition 5.03: The space (X_0, $Q(X_0)$) is called the pg-T_0–identification space of (X,τ), where X_0 is the set of equivalence classes of $\Re$ and $Q(X_0)$ is the decomposition topology on X_0.
Let P_X: $(X,\tau)\to(X_0, Q(X_0))$ denote the natural map

Lemma 5.04: *If $x\in X$ and $A\subset X$, then $x\in$ pgclA iff every* pg-*open set containing x intersects A.*

Theorem 5.05: *The natural map $P_X:(X,\tau)\to(X_0, Q(X_0))$ is closed, open and $P_X^{-1}(P_X(O)) = O$ for all $O\in PO(X,\tau)$ and $(X_0, Q(X_0))$ is* pg-T_0

Proof: Let $O\in PO(X,\tau)$ and let $C\in P_X(O)$. Then there exists $x\in O$ such that $P_X(x) = C$. If $y\in C$, then *pgcl*{y} = *pgcl*{x}, which, by lemma, implies $y\in O$. Since $\tau\subset PO(X,\tau)$, then $P_X^{-1}(P_X(U)) = U$ for all $U\in\tau$, which implies P_X is closed and open.

Let G, $H\in X_0$ such that $G\neq H$; let $x\in G$ and $y\in H$. Then *pgcl*{x} $\neq$ *pgcl*{y}, which implies $x\notin$*pgcl*{y} or $y\notin$*pgcl*{x}, say $x\notin$*pgcl*{y}. Since P_X is continuous and open, then $G\in A = P_X\{X\sim$*pgcl*{y}$\}\notin PO(X_0, Q(X_0))$ and $H\notin A$

Theorem 5.06: *The following are equivalent:*
(i) X is pgR_0 (ii) X_0 = {pgcl{x}: $x\in X$} and (iii) $(X_0, Q(X_0))$ is pgT_1

Proof: (i) $\Rightarrow$ (ii) Let $C\in X_0$, and let $x\in C$. If $y\in C$, then $y\in$*pgcl*{y} = *pgcl*{x}, which implies $C\in$*pgcl*{x}. If $y\in$*pgcl*{x}, then $x\in$*pgcl*{y}, since, otherwise, $x\in X\sim$*pgcl*{y}$\in PO(X,\tau)$ which implies *pgcl*{x}$\subset X\sim$*pgcl*{y}, which is a contradiction. Thus, if $y\in$*pgcl*{x}, then $x\in$*pgcl*{y}, which implies *pgcl*{y} = *pgcl*{x} and $y\in C$. Hence X_0 = {*pgcl*{x}: $x\in X$}

(ii)$\Rightarrow$(iii) Let $A\neq B\in X_0$. Then there exists x, $y\in X$ such that A = *pgcl*{x}; B = *pgcl*{y}, and *pgcl*{x}$\cap$*pgcl*{y} = ϕ. Then $A\in C = P_X(X\sim$*pgcl*{y}$)\in PO(X_0, Q(X_0))$ and $B\notin C$. Thus $(X_0, Q(X_0))$ is pg-T_1

(iii) $\Rightarrow$ (i) Let $x\in U\in\alpha GO(X)$. Let $y\notin U$ and C_x, $C_y\in X_0$ containing x and y respectively. Then $x\notin$ *pgcl*{y}, which implies $C_x\neq C_y$ and there exists pg-open set A such that $C_x\in A$ and $C_y\notin A$. Since P_X is continuous and open, then $y\in B = P_X^{-1}(A)\in x\in PGO(X)$ and $x\notin B$, which implies $y\notin$*pgcl*{x}. Thus *pgcl*{x}$\subset U$. This is true for all *pgcl*{x} implies $\cap$*pgcl*{x}$\subset U$. Hence X is pg-R_0

Theorem 5.07: *(X, τ) is* pg-R_1 *iff $(X_0, Q(X_0))$ is* pg-T_2

The proof is straight forward from theorems 5.05 and 5.06 and is omitted

Theorem 5.08: *X is* pg-T_i; *i = 0,1,2. iff there exists a* pg-*continuous, almost–open, 1–1 function from (X, τ) into a* pg-T_i *space ; i = 0,1,2. respectively.*

Theorem 5.09: *If f: $(X,\tau)\to(Y,\sigma)$ is* pg-*continuous,* pg-*open, and x, $y\in X$ such that pgcl{x} = pgcl{y}, then pgcl{f(x)} = pgcl{f(y)}.*

Theorem 5.10: *The following are equivalent*
(i) (X, τ) is pg-T_0
(ii) Elements of X_0 are singleton sets and
(iii)There exists a pg-*continuous,* pg-*open, 1–1 function f: $(X,\tau)\to(Y,\sigma)$, where (Y,σ) is* pg-T_0

Proof: (i) is equivalent to (ii) and (i) $\Rightarrow$ (iii) are straight forward and is omitted.

(iii) $\Rightarrow$ (i) Let x, $y\in X$ such that $f(x)\neq f(y)$, which implies *pgcl*{f(x)} $\neq$ *pgcl*{f(y)}. Then by theorem 5.09, *pgcl*{x} $\neq$ *pgcl*{y}. Hence (X,τ) is pg-T_0

Corollary 5.11: *A space* (X, τ) *is* pg-T_i ; $i = 1,2$ *iff* (X, τ) *is* pg-T_{i-1} ; $i = 1,2$, *respectively, and there exists a* pg-*continuous ,* pg-*open, 1–1 function* $f: (X, \tau)$ *into a* pg-T_0 *space.*

Definition 5.04: $f:X\to Y$ is point–pg-closure 1–1 iff for x, y∈X such that $pgcl\{x\} \neq pgcl\{y\}$, $pgcl\{f(x)\} \neq pgcl\{f(y)\}$.

Theorem 5.12:
(i)If $f: (X, \tau)\to(Y, \sigma)$ *is point–* pg-*closure 1–1 and* (X, τ) *is* pg-T_0, *then* f *is 1–1*
(ii)If $f: (X, \tau)\to(Y, \sigma)$, *where* (X, τ) *and* (Y, σ) *are* pg-T_0 *then* f *is point–* pg-*closure 1–1 iff* f *is 1–1*

The following result can be obtained by combining results for pg-T_0– identification spaces, pg-induced functions and pg-T_i spaces; i = 1,2.

Theorem 5.13: *X is* pg-R_i ; $i = 0,1$ *iff there exists a* pg-*continuous , almost–open point–* pg-*closure 1–1 function* $f: (X, \tau)$ *into a* pg-R_i *space;* $i = 0,1$ *respectively.*

6. pg-**Normal; Almost** pg-**normal and Mildly** pg-**normal spaces**

Definition 6.1: A space X is said to be pg-normal if for any pair of disjoint closed sets F_1 and F_2 , there exist disjoint pg-open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.
Example 4: Let X = {a, b, c} and τ = {ϕ, {a}, {b, c}, X}. Then X is pg-normal.

Example 5: Let X = {a, b, c, d} and τ = {ϕ, {b, d},{a, b, d},{b, c, d}, X}. Then X is not *pg*-normal and is not normal.

Example 6: Let X = {a, b, c, d} with τ = {ϕ, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, c}, {a, b, d}, X} is *pg*-normal, normal and almost normal.

We have the following characterization of pg-normality.

Theorem 6.1: For a space X the following are equivalent:
(i) *X* is pg-normal.
(ii) For every pair of open sets *U* and *V* whose union is *X*, there exist pg-closed sets *A* and *B* such that $A\subset U$, $B \subset V$ and $A\cup B = X$.
(iii) For every closed set *F* and every open set *G* containing *F*, there exists a pg-open set *U* such that $F\subset U\subset pgcl(U)\subset G$.

Proof: (i)⇒(ii): Let *U* and *V* be a pair of open sets in a pg-normal space *X* such that $X = U\cup V$. Then $X–U$, $X–V$ are disjoint closed sets. Since X is pg-normal there exist disjoint pg-open sets U_1 and V_1 such that $X–U\subset U_1$ and $X-V\subset V_1$.

Let $A = X–U_1$, $B = X–V_1$. Then *A* and *B* are pg-closed sets such that $A\subset U$, $B\subset V$ and $A\cup B = X$.

(b) ⇒(c): Let *F* be a closed set and *G* be an open set containing *F*. Then $X–F$ and *G* are open sets whose union is *X*. Then by (b), there exist pg-closed sets W_1 and W_2 such that $W_1 \subset X–F$ and $W_2 \subset G$ and $W_1\cup W_2 = X$. Then $F\subset X–W_1$, $X–G \subset X–W_2$ and $(X–W_1)\cap(X–W_2) = \phi$. Let $U = X–W_1$ and $V= X–W_2$. Then *U* and *V* are disjoint pg-open sets such that $F\subset U\subset X–V\subset G$. As $X–V$ is pg-closed set, we have $pgcl(U) \subset X–V$ and $F\subset U\subset pgcl(U)\subset G$.

(c) ⇒ (a): Let F_1 and F_2 be any two disjoint closed sets of *X*. Put $G = X–F_2$, then $F_1\cap G = \phi$. $F_1\subset G$ where *G* is an open set. Then by (c), there exists a pg-open set *U* of *X* such that $F_1 \subset U \subset pgcl(U) \subset G$. It follows that $F_2 \subset X–pgcl(U) = V$, say, then *V* is pg-open and $U\cap V = \phi$. Hence F_1 and F_2 are separated by pg-open sets *U* and *V*. Therefore *X* is pg-normal.

Theorem 6.2: A regular open subspace of a pg-normal space is pg-normal.

Example 7: Let X = {a, b, c, d} with τ = {ϕ, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, c}, {a, b, d}, X} is pg-normal and pg-regular.

However we observe that every pg-normal pg-R_0 space is pg-regular.

Definition 6.2: A function $f:X \to Y$ is said to be almost –pg-irresolute if for each x in X and each pg-neighborhood V of $f(x)$, $pgcl(f^{-1}(V))$ is a pg-neighborhood of x.
Clearly every pg-irresolute map is almost pg-irresolute.
The Proof of the following lemma is straightforward and hence omitted.

Lemma 6.1: f is almost pg-irresolute iff $f^{-1}(V) \subset$ pg-int($pgcl(f^{-1}(V))$) for every $V \in PGO(Y)$.

Lemma 6.2: f is almost pg-irresolute iff $f(pgcl(U)) \subset pgcl(f(U))$ for every $U \in PGO(X)$.

Proof: Let $U \in PGO(X)$.Suppose $y \notin pgcl(f(U))$. Then there exists $V \in PGO(y)$ such that $V \cap f(U) = \phi$. Hence $f^{-1}(V) \cap U = \phi$. Since $U \in PGO(X)$, we have pg-int($pgcl(f^{-1}(V))$) $\cap\, pgcl(U) = \phi$. Then by lemma 6.1, $f^{-1}(V) \cap pgcl(U) = \phi$ and hence $V \cap f(pgcl(U)) = \phi$. This implies that $y \notin f(pgcl(U))$.

Conversely, if $V \in PGO(Y)$, then $W = X- pgcl(f^{-1}(V)) \in PGO(X)$. By hypothesis, $f(pgcl(W)) \subset pgcl(f(W))$ and hence X- pg-int($pgcl(f^{-1}(V))$) $= pgcl(W) \subset f^{-1}(pgcl(f(W))) \subset f(pgcl[f(X-f^{-1}(V))]) \subset f^{-1}[pgcl(Y-V)] = f^{-1}(Y-V) = X-f^{-1}(V)$.

Therefore, $f^{-1}(V) \subset$ pg-int($pgcl(f^{-1}(V))$). By lemma 6.1, f is almost pg-irresolute.

Now we prove the following result on the invariance of pg-normality.

Theorem 6.3: If f is an M-pg-open continuous almost pg-irresolute function from a pg-normal space X onto a space Y, then Y is pg-normal.

Proof: Let A be a closed subset of Y and B be an open set containing A. Then by continuity of f, $f^{-1}(A)$ is closed and $f^{-1}(B)$ is an open set of X such that $f^{-1}(A) \subset f^{-1}(B)$. As X is pg-normal, there exists a pg-open set U in X such that $f^{-1}(A) \subset U \subset pgcl(U) \subset f^{-1}(B)$. Then $f(f^{-1}(A)) \subset f(U) \subset f(pgcl(U)) \subset f(f^{-1}(B))$. Since f is M-pg-open almost pg-irresolute surjection, we obtain $A \subset f(U) \subset pgcl(f(U)) \subset B$. Then again by Theorem 6.1 the space Y is pg-normal.

Lemma 6.3: A mapping f is M-pg-closed if and only if for each subset B in Y and for each pg-open set U in X containing $f^{-1}(B)$, there exists a pg-open set V containing B such that $f^{-1}(V) \subset U$.

Theorem 6.4: If f is an M-pg-closed continuous function from a pg-normal space onto a space Y, then Y is pg-normal.

Proof of the theorem is routine and hence omitted.

Now in view of lemma 2.2 [9] and lemma 6.3, we prove that the following result.

Theorem 6.5: If f is an M-pg-closed map from a weakly Hausdorff pg-normal space X onto a space Y such that $f^{-1}(y)$ is S-closed relative to X for each $y \in Y$, then Y is pg-T_2.

Proof: Let y_1 and y_2 be any two distinct points of Y. Since X is weakly Hausdorff, $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are disjoint closed subsets of X by lemma 2.2 [9]. As X is pg-normal, there exist disjoint pg-open sets V_1 and V_2 such that $f^{-1}(y_i) \subset V_i$, for i = 1,2. Since f is M-pg-closed, there exist pg-open sets U_1 and U_2 containing y_1 and y_2 such that $f^{-1}(U_i) \subset V_i$ for i = 1,2. Then it follows that $U_1 \cap U_2 = \phi$. Hence Y is pg-T_2.

Theorem 6.6: For a space X we have the following:
(a) If X is normal then for any disjoint closed sets A and B, there exist disjoint pg-open sets U, V such that $A \subset U$ and $B \subset V$;
(b) If X is normal then for any closed set A and any open set V containing A, there exists an pg-open set U of X such that $A \subset U \subset pgcl(U) \subset V$.

Definition 6.2: X is said to be almost pg-normal if for each closed set A and each regular closed set B such that $A \cap B = \phi$, there exist disjoint pg-open sets U and V such that $A \subset U$ and $B \subset V$.

Clearly, every pg-normal space is almost pg-normal, but not conversely in general.

Now, we have characterization of almost pg-normality in the following.

Theorem 6.7: For a space X the following statements are equivalent:
(i) X is almost pg-normal
(ii) For every pair of sets U and V , one of which is open and the other is regular open whose union is X, there exist pg-closed sets G and H such that $G \subset U$,$H \subset V$ and $G \cup H = X$.
(iii) For every closed set A and every regular open set B containing A, there is a pg-open set V such that $A \subset V \subset pgcl(V) \subset B$.

Proof: (a)$\Rightarrow$(b) Let U be an open set and V be a regular open set in an almost pg-normal space X such that $U \cup V = X$. Then (X-U) is closed set and (X-V) is regular closed set with $(X-U) \cap (X-V) = \phi$. By almost pg-normality of X, there exist disjoint pg-open sets U_1 and V_1 such that $X-U \subset U_1$ and $X-V \subset V_1$. Let $G = X- U_1$ and $H = X-V_1$. Then G and H are pg-closed sets such that $G\subset U$, $H\subset V$ and $G\cup H = X$.

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are obvious.
One can prove that almost pg-normality is also regular open hereditary.

Almost pg-normality does not imply almost pg-regularity in general. However, we observe that every almost pg-normal pg-R_0 space is almost pg-regular.

Theorem 6.8: Every almost regular, *pg*-compact space X is almost pg-normal.

Recall that a function f: $X\rightarrow Y$ is called rc-continuous if inverse image of regular closed set is regular closed.

Now, we state the invariance of almost pg-normality in the following.

Theorem 6.9: If f is continuous M-pg-open rc-continuous and almost pg-irresolute surjection from an almost pg-normal space X onto a space Y, then Y is almost pg-normal.

Definition 6.3: A space X is said to be mildly pg-normal if for every pair of disjoint regular closed sets F_1 and F_2 of X, there exist disjoint pg-open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

Example 8: Let X = {a, b, c, d} with τ = {ϕ, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, c}, {a, b, d}, X} is Mildly *pg*-normal.

We have the following characterization of mild pg-normality.

Theorem 6.10: For a space X the following are equivalent.
(i) X is mildly pg-normal.
(ii) For every pair of regular open sets U and V whose union is X, there exist pg-closed sets G and H such that $G \subset U$, $H \subset V$ and $G\cup H = X$.
(iii) For any regular closed set A and every regular open set B containing A, there exists a pg-open set U such that $A\subset U\subset pgcl(U)\subset B$.
(iv) For every pair of disjoint regular closed sets, there exist pg-open sets U and V such that $A\subset U$, $B\subset V$ and $pgcl(U)\cap pgcl(V) = \phi$.

This theorem may be proved by using the arguments similar to those of Theorem 6.7.

Also, we observe that mild pg-normality is regular open hereditary.

Definition 6.4: A space X is weakly pg-regular if for each point x and a regular open set U containing {x}, there is a pg-open set V such that $x\in V \subset clV \subset U$.

Example 9: Let X = {a, b, c} and τ = {ϕ, {b},{a, b},{b, c}, X}. Then X is weakly *pg*-regular.

Example 10: Let X = {a, b, c} and τ = {ϕ, {a},{b},{a, b}, X}. Then X is not weakly *pg*-regular.

Theorem 6.11: If $f : X \rightarrow Y$ is an M-pg-open rc-continuous and almost pg-irresolute function from a mildly pg-normal space X onto a space Y, then Y is mildly pg-normal.

Proof: Let A be a regular closed set and B be a regular open set containing A. Then by rc-continuity of f, $f^{-1}(A)$ is a regular closed set contained in the regular open set $f^{-1}(B)$. Since X is mildly pg-normal, there exists a pg-open set V such that $f^{-1}(A) \subset V\subset pgcl(V) \subset f^{-1}(B)$ by Theorem 6.10. As f is M-pg-open and almost pg-irresolute surjection, it follows that $f(V)\in PGO(Y)$ and $A\subset f(V) \subset pgcl(f(V))\subset B$. Hence Y is mildly pg-normal.

Theorem 6.12: If f: $X \rightarrow Y$ is rc-continuous, M-pg-closed map from a mildly pg-normal space X onto a space Y, then Y is mildly pg-normal.

7. pg-**US SPACES:**

Definition 7.1:A sequence $<x_n>$ is said to be *pg*-converges to a point x of X, written as $<x_n> \rightarrow^{pg} x$ if $<x_n>$ is eventually in every *pg*-open set containing x.

Clearly, if a sequence $<x_n>$ *r*-converges to a point x of X, then $<x_n>$ *pg*-converges to x.

Definition 7.2: X is said to be *pg*-US if every sequence $<x_n>$ in X *pg*-converges to a unique point.

Definition 7.3: A set F is sequentially *pg*-closed if every sequence in F *pg*-converges to a point in F.

Definition 7.4: A subset G of a space X is said to be sequentially *pg*-compact if every sequence in G has a subsequence which *pg*-converges to a point in G.

Definition 7.5: A point y is a *pg*-cluster point of sequence $<x_n>$ iff $<x_n>$ is frequently in every *pg*-open set containing x. The set of all *pg*-cluster points of $<x_n>$ will be denoted by *pg*-cl(x_n).

Definition 7.6: A point y is *pg*-side point of a sequence $<x_n>$ if y is a *pg*-cluster point of $<x_n>$ but no subsequence of $<x_n>$ *pg*-converges to y.

Definition 7.7: A space X is said to be
(i) *pg*-S_1 if it is *pg*-US and every sequence $<x_n>$ *pg*-converges with subsequence of $<x_n>$ *pg*-side points.
(ii) *pg*-S_2 if it is *pg*-US and every sequence $<x_n>$ in X *pg*-converges which has no *pg*-side point.

Using sequentially continuous functions, we define sequentially *pg*-continuous functions.

Definition 7.8: A function *f* is said to be sequentially *pg*-continuous at $x \in X$ if $f(x_n) \to^{pg} f(x)$ whenever $<x_n> \to^{pg} x$. If *f* is sequentially *pg*-continuous at all $x \in X$, then *f* is said to be sequentially *pg*-continuous.

Theorem 7.1: We have the following:
(i) Every *pg*-T_2 space is *pg*-US.
(ii) Every *pg*-US space is *pg*-T_1.
(iii) X is *pg*-US iff the diagonal set is a sequentially *pg*-closed subset of X x X.
(iv) X is *pg*-T_2 iff it is both *pg*-R_1 and *pg*-US.
(v) Every regular open subset of a *pg*-US space is *pg*-US.
(vi) Product of arbitrary family of *pg*-US spaces is *pg*-US.
(vii) Every *pg*-S_2 space is *pg*-S_1 and Every *pg*-S_1 space is *pg*-US.

Theorem 7.2: In a *pg*-US space every sequentially *pg*-compact set is sequentially *pg*-closed.

Proof: Let X be *pg*-US space. Let Y be a sequentially *pg*-compact subset of X. Let $<x_n>$ be a sequence in Y. Suppose that $<x_n>$ *pg*-converges to a point in X-Y. Let $<x_{np}>$ be subsequence of $<x_n>$ that *pg*-converges to a point $y \in Y$ since Y is sequentially *pg*-compact. Also, let a subsequence $<x_{np}>$ of $<x_n>$ *pg*-converge to $x \in X\text{-}Y$. Since $<x_{np}>$ is a sequence in the *pg*-US space X, x = y. Thus, Y is sequentially *pg*-closed set.

Theorem 7.3: Let *f* and *g* be two sequentially *pg*-continuous functions. If Y is *pg*-US, then the set $A = \{x \mid f(x) = g(x)\}$ is sequentially *pg*-closed.

Proof: Let Y be *pg*-US and suppose that there is a sequence $<x_n>$ in A *pg*-converging to $x \in X$. Since *f* and *g* are sequentially *pg*-continuous functions, $f(x_n) \to^{pg} f(x)$ and $g(x_n) \to^{pg} g(x)$. Hence $f(x) = g(x)$ and $x \in A$. Therefore, A is sequentially *pg*-closed.

8. SEQUENTIALLY sub-pg-CONTINUITY:

In this section we introduce and study the concepts of sequentially sub-pg-continuity, sequentially nearly pg-continuity and sequentially pg-compact preserving functions and study their relations and the property of pg-US spaces.

Definition 8.1: A function *f* is said to be
(i) sequentially nearly pg-continuous if for each point $x \in X$ and each sequence $<x_n> \to^{pg} x$ in X, there exists a subsequence $<x_{nk}>$ of $<x_n>$ such that $<f(x_{nk})> \to^{pg} f(x)$.
(ii) sequentially sub-pg-continuous if for each point $x \in X$ and each sequence $<x_n> \to^{pg} x$ in X, there exists a subsequence $<x_{nk}>$ of $<x_n>$ and a point $y \in Y$ such that $<f(x_{nk})> \to^{pg} y$.
(iii) sequentially pg-compact preserving if *f*(K) is sequentially pg-compact in Y for every sequentially pg-compact set K of X.

Lemma 8.1: Every function *f* is sequentially sub-pg-continuous if Y is a sequentially pg-compact.

Proof: Let $<x_n> \to^{pg}$ x in X. Since Y is sequentially pg-compact, there exists a subsequence $\{f(x_{nk})\}$ of $\{f(x_n)\}$ pg-converging to a point $y \in Y$. Hence f is sequentially sub-pg-continuous.

Theorem 8.1: Every sequentially nearly pg-continuous function is sequentially pg-compact preserving.

Proof: Assume f is sequentially nearly pg-continuous and K any sequentially pg-compact subset of X. Let $<y_n>$ be any sequence in f (K). Then for each positive integer n, there exists a point $x_n \in K$ such that $f(x_n) = y_n$. Since $<x_n>$ is a sequence in the sequentially pg-compact set K, there exists a subsequence $<x_{nk}>$ of $<x_n>$ pg-converging to a point $x \in K$. By hypothesis, f is sequentially nearly pg-continuous and hence there exists a subsequence $<x_j>$ of $<x_{nk}>$ such that $f(x_j) \to^{pg} f(x)$. Thus, there exists a subsequence $<y_j>$ of $<y_n>$ pg-converging to $f(x) \in f(K)$. This shows that $f(K)$ is sequentially pg-compact set in Y.

Theorem 8.2: Every sequentially pre-continuous function is sequentially pg-continuous.

Proof: Let f be a sequentially pre-continuous and $<x_n> \to^{p} x \in X$. Then $<x_n> \to^{p}$ x. Since f is sequentially pre-continuous, $f(x_n) \to^{p} f(x)$. But we know that $<x_n> \to^{p}$ x implies $<x_n> \to^{pg}$ x and hence $f(x_n) \to^{pg} f(x)$ implies f is sequentially pg-continuous.

Theorem 8.3: Every sequentially pg-compact preserving function is sequentially sub-pg-continuous.

Proof: Suppose f is a sequentially pg-compact preserving function. Let x be any point of X and $<x_n>$ any sequence in X pg-converging to x. We shall denote the set $\{x_n \mid n= 1,2,3, \ldots\}$ by A and $K = A \cup \{x\}$. Then K is sequentially pg-compact since $(x_n) \to^{pg}$ x. By hypothesis, f is sequentially pg-compact preserving and hence $f(K)$ is a sequentially pg-compact set of Y. Since $\{f(x_n)\}$ is a sequence in $f(K)$, there exists a subsequence $\{f(x_{nk})\}$ of $\{f(x_n)\}$ pg-converging to a point $y \in f(K)$. This implies that f is sequentially sub-pg-continuous.

Theorem 8.4: A function f: $X \to Y$ is sequentially pg-compact preserving iff $f_{/K}: K \to f(K)$ is sequentially sub-pg-continuous for each sequentially pg-compact subset K of X.

Proof: Suppose f is a sequentially pg-compact preserving function. Then $f(K)$ is sequentially pg-compact set in Y for each sequentially pg-compact set K of X. Therefore, by Lemma 8.1 above, $f_{/K}: K \to f(K)$ is sequentially pg-continuous function.

Conversely, let K be any sequentially pg-compact set of X. Let $<y_n>$ be any sequence in $f(K)$. Then for each positive integer n, there exists a point $x_n \in K$ such that $f(x_n) = y_n$. Since $<x_n>$ is a sequence in the sequentially pg-compact set K, there exists a subsequence $<x_{nk}>$ of $<x_n>$ pg-converging to a point $x \in K$. By hypothesis, $f_{/K}: K \to f(K)$ is sequentially sub-pg-continuous and hence there exists a subsequence $<y_{nk}>$ of $<y_n>$ pg-converging to a point $y \in f(K)$.This implies that $f(K)$ is sequentially pg-compact set in Y. Thus, f is sequentially pg-compact preserving function.

The following corollary gives a sufficient condition for a sequentially sub-pg-continuous function to be sequentially pg-compact preserving.

Corollary 8.1: If f is sequentially sub-pg-continuous and f (K) is sequentially pg-closed set in Y for each sequentially pg-compact set K of X, then f is sequentially pg-compact preserving function.

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