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SOME FIXED POINT THEOREMS IN TWO METRIC SPACES

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ABSTRACT

In this paper we prove some fixed point theorems for generalized contraction mappings in two complete metric spaces. Here we extend some results due to B. Fisher.

Key words and Phrases: *fixed point, common fixed point and complete metric space.*

AMS Mathematics Subject Classification: *47H10, 54H25.*

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1. INTRODUCTION.

Some authors proved many kinds of fixed point theorems for contractive type mappings and non-expansive mappings ([1]-[4]). In [5] and [6], B. Fisher proved some theorems in two complete metric spaces as follows:

Theorem 1.1: [5] Let (X, d) and (Y, e) be complete metric spaces. If T is a mapping from X into Y and S is a mapping from Y into X, satisfying the following conditions:

$$e(Tx, TSy) \leq c \,.\, \max\{d(x, Sy), e(y, Tx), e(y,TSy)\}$$
$$d(Sy, STx) \leq c \,.\, \max\{e(y, Tx), d(x, Sy), d(x, STx)\}$$

for all x in X and y in Y. where $0 \leq c < 1$, then ST have a unique fixed point z in X and TS has a unique fixed point w in Y. Further, Tz = w and Sw = z.

In this paper we prove some fixed point theorems in two complete metric spaces. Our aim is to extend the results of B. Fisher [4] and [5]. The following definitions are necessary for the present study.

Definition1.2: A sequence $\{x_n\}$ in a metric space (X, d) is said to be convergent to a point $x \in X$ if given $\in > 0$ there exists a positive integer n_0 such that $d(x_n,x) < \in$ for all $n \geq n_0$.

Definition1.3: A sequence $\{x_n\}$ in a metric space (X, d) is said to be a Cauchy sequence in X if given $\in>0$ there exists a positive integer n_0 such that $d(x_m, x_n) < \in$ for all $m, n \geq n_0$.

Definition1.4: A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition1.5: Let X be a non-empty set and f: X → X be a map. An element x in X is called a fixed point of X f(x)= x .

Definition1.6: Let X be a non-empty set and f, g: X → X be two maps. An element x in X is called a common fixed point of f and g if f(x) = g(x) = x.

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2 MAIN RESULTS:

Theorem2.1: Let (X, d) and (Y, e) be complete metric spaces. If T is a mapping from X into Y and S is a mapping from Y into X satisfying the following conditions:

$$e(Tx, TSy) \le c_1.\max\{d(x, Sy), e(y, Tx) + e(y, TSy)\} \quad (1)$$

$$d(Sy, STx) \le c_2.\max\{d(x, Sy) + d(x, STx), e(y, Tx)\} \quad (2)$$

for all x in X and y in Y where $0 \le c_1 < 1$ and $0 \le c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further Tz = w and Sw = z.

Proof: Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X and a sequence $\{y_n\}$ in Y, as follows:

$$x_n = (ST)^n x_0,\ y_n = T(x_{n-1}) \text{ for } n = 1,2,\dots .$$

We have

$$\begin{aligned} d(x_n, x_{n+1}) &= d((ST)^n x_0, (ST)^{n+1} x_0)) \\ &= d(S(T(ST)^{n-1} x_0, ST(ST)^n x_0) \\ &= d(ST(x_{n-1}), STx_n) \\ &= d(Sy_n, STx_n) \\ &\le c_2.\max\{d(x_n, Sy_n) + d(x_n, STx_n), e(y_n, Tx_n)\} \quad \text{(since by (2))} \\ &= c_2.\max\{d(x_n, x_n) + d(x_n, x_{n+1}), e(y_n, y_{n+1})\} \\ &= c_2.\max\{d(x_n, x_{n+1}), e(y_n, y_{n+1})\} \\ &\le c_2.\, e(y_n, y_{n+1}) \end{aligned}$$

Now

$$\begin{aligned} e(y_n, y_{n+1}) &= e(Tx_{n-1}, Tx_n) \\ &= e(Tx_{n-1}, TSy_n) \\ &\le c_1.\max\{d(x_{n-1}, Sy_n), e(y_n, Tx_{n-1}) + e(y_n, TSy_n)\} \quad \text{(since by (1))} \\ &= c_1.\max\{d(x_{n-1}, x_n), e(y_n, y_n) + e(y_n, y_{n+1})\} \\ &\le c_1.\, d(x_{n-1}, x_n) \end{aligned}$$

Hence

$$\begin{aligned} d(x_n, x_{n+1}) &\le c_1c_2.\, d(x_{n-1}, x_n) \\ &\ \vdots \\ &\le (c_1c_2)^n d(x_0, x_1) \to 0 \text{ as } n\to\infty \qquad (\text{since } 0 \le c_1c_2 < 1) \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, it converges to a point z in X. Similarly, we can prove that the sequence $\{y_n\}$ is also a Cauchy sequence in (Y, e). Since (Y, e) is complete, it converges to a point w in Y.

Now we prove Tz = w

Suppose $Tz \ne w$.

We have

$$\begin{aligned} e(Tz, w) &= \lim_{n\to\infty} e(Tz, y_{n+1}) \\ &= \lim_{n\to\infty} e(Tz, TSy_n) \\ &\le \lim_{n\to\infty} c_1.\max\{d(z, Sy_n), e(y_n, Tz) + e(y_n, TSy_n)\} \\ &= \lim_{n\to\infty} c_1.\max\{d(z, x_n), e(y_n, Tz) + e(y_n, y_{n+1})\} \\ &\le c_1.e(Tz, w) \\ &< e(Tz, w) \quad (\text{since } 0 \le c_1 < 1), \text{ which is a contradiction.} \end{aligned}$$

Thus Tz = w.

Now we prove Sw = z.

Suppose Sw $\neq$ z.

We have

$$\begin{aligned} d(Sw, z) &= \lim_{n\to\infty} d(Sw, x_{n+1}) \\ &= \lim_{n\to\infty} d(Sw, STx_n) \\ &\leq \lim_{n\to\infty} c_2.\max\{d(x_n, Sw) + d(x_n, STx_n), e(w, Tx_n)\} \\ &= \lim_{n\to\infty} c_2.\max\{d(x_n, Sw) + d(x_n, x_{n+1}), e(w, y_{n+1})\} \\ &\leq c_2.\, d(Sw, z) \\ &< d(Sw, z) \ \text{(since } 0 \leq c_2 < 1\text{), which is a contradiction.} \end{aligned}$$

Thus Sw = z.

We have STz = Sw = z and TSw = Tz = w. Thus the point z is a fixed point of ST in X and the point w is a fixed point of TS in Y.

Uniqueness: Let z′ $\neq$ z be another fixed point of ST in X.

We have

$$\begin{aligned} d(z, z') &= d(STz, STz') \\ &\leq c_2.\max\{d(z', STz) + d(z', STz'), e(Tz, Tz')\} \\ &= c_2.\max\{d(z', z), e(Tz,Tz')\} \\ &\leq c_2.e(Tz, Tz') \end{aligned}$$

Also we have

$$\begin{aligned} e(Tz, Tz') &= e(Tz, TSTz') \\ &\leq c_1.\max\{d(z, STz'), e(Tz',Tz) + e(Tz', TSTz')\} \\ &= c_1.\max\{d(z, z'), e(Tz', Tz)\} \\ &\leq c_1.d(z, z') \end{aligned}$$

Hence

$$d(z, z') \leq c_1c_2.d(z, z') < d(z, z') \ \text{(since } c_1c_2 < 1\text{), which is a contradiction.}$$

Thus z = z′.

So the point z is a unique fixed point of ST. Similarly, we prove the point w is also a unique fixed point of TS. This completes the proof

Remark 2.2: If (X, d) and (Y, e) are the same metric spaces, then by the above theorem 2.1, we get the following theorem, as corollary.

Corollary2.3: Let (X, d) be a complete metric space. If S and T are mappings from X into itself satisfying the following conditions:

$$d(Tx, TSy) \leq c_1.\max\{d(x, Sy), d(y, Tx) + d(y, TSy)\}$$
$$d(Sy, STx) \leq c_2.\max\{d(x, Sy) + d(x, STx), d(y, Tx)\}$$

for all x, y in X where $0 \leq c_1, c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in X. Further, Tz = w and Sw = z and if z = w, then z is the unique common fixed point of S and T.

Theorem2.4: Let (X, d) and (Y, e) be two complete metric spaces. If T is a mapping from X into Y and S is a mapping from Y into X satisfying the following conditions

$$e(Tx, TSy) \leq c_1.\max\{d(x, Sy), e(y, Tx), e(y, Tx) + e(y, TSy)\} \quad (1)$$
$$d(Sy, STx) \leq c_2.\max\{e(y, Tx), d(x, Sy), d(x, Sy) + d(x, STx)\} \quad (2)$$

for all x in X and y in Y where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further Tz = w and Sw = z.

Proof: Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X and a sequence $\{y_n\}$ in Y, as follows:

$$x_n = (ST)^n x_0, \ y_n = T(x_{n-1}) \text{ for } n = 1,2,\dots .$$

Now we have

$$\begin{aligned}
d(x_n, x_{n+1}) &= d((ST)^n x_0, (ST)^{n+1} x_0)) \\
&= d(S(T(ST)^{n-1} x_0, ST(ST)^n x_0) \\
&= d(ST(x_{n-1}), STx_n) \\
&= d(Sy_n, STx_n) \\
&\le c_2.\max\{e(y_n, Tx_n), d(x_n, Sy_n), d(x_n, Sy_n) + d(x_n, STx_n)\} \\
&= c_2.\max\{e(y_n, y_{n+1}), d(x_n, x_n), d(x_n, x_n) + d(x_n, x_{n+1})\} \\
&= c_2.\max\{e(y_n, y_{n+1}), 0, d(x_n, x_{n+1})\} \\
&\le c_2.\ e(y_n, y_{n+1})
\end{aligned}$$

Now

$$\begin{aligned}
e(y_n, y_{n+1}) &= e(Tx_{n-1}, Tx_n) \\
&= e(Tx_{n-1}, TSy_n) \\
&\le c_1.\max\{d(x_{n-1},Sy_n), e(y_n,Tx_{n-1}), e(y_n,Tx_{n-1}) + e(y_n, Sy_n)\} \\
&= c_1.\max\{d(x_{n-1}, x_n), e(y_n, y_n), e(y_n, y_n) + e(y_n, y_{n+1})\} \\
&\le c_2.\ d(x_{n-1}, x_n)
\end{aligned}$$

Hence

$$\begin{aligned}
d(x_n, x_{n+1}) &\le c_1c_2.\ d(x_{n-1}, x_n) \\
&\ \vdots \\
&\le (c_1c_2)^n d(x_0, x_1) \to 0 \text{ as } n\to\infty \quad (\text{since } 0\le c_1c_2 <1)
\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, it converges to a point z in X. Similarly, we can prove that the sequence $\{y_n\}$ is also a Cauchy sequence in (Y, e). Since (Y, e) is complete, it converges to a point w in Y.

Now we prove $Tz = w$.

Suppose $Tz \neq w$.

We have

$$\begin{aligned}
e(Tz, w) &= \lim_{n\to\infty} e(Tz, y_{n+1}) \\
&= \lim_{n\to\infty} e(Tz, TSy_n) \\
&\le \lim_{n\to\infty} c_1.\max\{d(z, Sy_n),e(y_n, Tz), e(y_n, Tz) + e(y_n, TSy_n)\} \\
&= \lim_{n\to\infty} c_1.\max\{d(z, x_n), e(y_n, Tz), e(y_n, Tz) + e(y_n, y_{n+1})\} \\
&= c_1.\ \max\{d(z, z), e(w, Tz), e(w, Tz) + e(w,w)\} \\
&= c_1.\ \max\{0, e(w, Tz), e(w, Tz)\} \\
&< e(w, Tz) \quad (\text{since } 0 \le c_1 < 1) \text{ ,which is a contradiction.}
\end{aligned}$$

Thus $Tz = w$.

Now we prove $Sw = z$.

Suppose $Sw \neq z$.

We have

$$\begin{aligned}
d(Sw, z) &= d(Sw, x_{n+1}) \\
&= \lim_{n\to\infty} d(Sw, STx_n) \\
&\le \lim_{n\to\infty} c_2.\max\{e(w,Tx_n), d(x_n,w), d(x_n, Sw) + d(x_n, STx_n)\} \\
&= \lim_{n\to\infty} c_2.\ \max\{e(w, y_{n+1}), d(x_n, Sw), d(x_n, Sw) + d(x_n, x_{n+1})\} \\
&< d(Sw, z) \quad (\text{since } 0 \le c_2 < 1) \text{ ,which is a contradiction.}
\end{aligned}$$

Thus Sw = z.

We have STz = Sw = z and TSw = Tz = w. Thus z is a fixed point of ST in X and the point w is a fixed point of TS in Y.

Uniqueness: Let $z' \neq z$ be another fixed point of ST in X.

We have

$$\begin{aligned} d(z', z) &= d(STz', STz) \\ &\leq c_2.\max\{e(Tz',Tz), d(z, STz'), d(z, STz') + d(z. STz)\} \\ &\leq c_2.\max\{e(Tz',Tz), d(z, z'), d(z, z')\} \\ &\leq c_2.e(Tz',Tz) \end{aligned}$$

Now

$$\begin{aligned} e(Tz',Tz) &= e(Tz', TSTz) \\ &\leq c_1.\max\{d(z', STz), e(Tz, Tz'), e(Tz, Tz') + e(Tz, TSTz)\} \\ &= c_1.\max\{d(z', z), e(Tz, Tz'), e(Tz, Tz')\} \\ &\leq c_1.d(z',z) \end{aligned}$$

Hence

$d(z',z) \leq c_1c_2.d(z',z) < d(z',z)$ (since $c_1c_2 <1$),which is a contradiction.

Thus $z = z'$.

So the point z is a unique fixed point of ST. Similarly, we prove the point w is also a unique point of TS. This completes the proof

Remark2.5: If (X, d) and (Y, e) are the same metric spaces, then by the above theorem 2.4, we get the following theorem as corollary.

Corollary2.6: Let (X, d) be a complete metric space. If S and T are mappings from X into itself satisfying the following conditions:

$$d(Tx, TSy) \leq c_1.\max\{d(x, Sy), d(y, Tx) ,d(y, Tx) + d(y, TSy)\}$$
$$d(Sy, STx) \leq c_2.\max\{d(y, Tx), d(x, Sy), d(x, Sy) + d(x, STx)\}$$

for all x, y in X where $0\leq c_1, c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in X. Further, Tz = w and Sw = z and if z = w, then z is the unique common fixed point of S and T.

Theorem2.7: Let (X, d) and (Y, e) be complete metric spaces. If T is a mapping from X into Y and S is a mapping from Y into X satisfying the following conditions

$$e(Tx, TSy) \leq c_1.\max\{d(x, Sy), e(y, Tx), e(y, TSy), d(x, STx)\} \quad (1)$$
$$d(Sy, STx) \leq c_2.\max\{e(y, Tx), d(x, Sy), d(x, STx), e(Tx, TSy)\} \quad (2)$$

for all x in X and y in Y where $0\leq c_1 < 1$and $0 \leq c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further Tz = w and Sw = z.

Proof: Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X and a sequence $\{y_n\}$ in Y, as follows:

$$x_n = (ST)^n x_0,\ y_n = T(x_{n-1}) \text{ for } n = 1,2,\dots .$$

We have

$$\begin{aligned} d(x_n, x_{n+1}) &= d((ST)^n x_0, (ST)^{n+1} x_0)) \\ &= d(S(T(ST)^{n-1} x_0, ST(ST)^n x_0) \\ &= d(ST(x_{n-1}), STx_n) \\ &= d(Sy_n, STx_n) \\ &\leq c_2.\max\{e(y_n, Tx_n), d(x_n, Sy_n), d(x_n, STx_n), e(Tx_n, TSy_n)\} \\ &= c_2.\max\{e(y_n, y_{n+1}), d(x_n, x_n), d(x_n, x_{n+1}), e(y_{n+1}, y_{n+1})\} \qquad \leq c_2.e(y_n, y_{n+1}) \end{aligned}$$

Now

$$
\begin{aligned}
e(y_n, y_{n+1}) &= e(Tx_{n-1}, Tx_n) \\
&= e(Tx_{n-1}, TSy_n) \\
&\le c_1.\max\{d(x_{n-1}, Sy_n), e(y_n, Tx_{n-1}), e(y_n, TSy_n), d(x_{n-1}, STx_{n-1})\} \\
&= c_1.\max\{d(x_{n-1}, x_n), e(y_n, y_n), e(y_n, y_{n+1}), d(x_{n-1}, x_n)\} \\
&\le c_1.\, d(x_{n-1}, x_n)
\end{aligned}
$$

Hence

$$
\begin{aligned}
d(x_n, x_{n+1}) &\le c_1c_2.\, d(x_{n-1}, x_n) \\
&\vdots \\
&\le (c_1c_2)^n\, d(x_0, x_1) \to 0 \text{ as } n\to\infty \quad (\text{since } 0 \le c_1c_2 < 1)
\end{aligned}
$$

Thus $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, it converges to a point z in X. Similarly, we can prove that the sequence $\{y_n\}$ is also a Cauchy sequence in (Y, e). Since (Y, e) is complete, it converges to a point w in Y.

Now we prove Tz = w.

Suppose $Tz \neq w$.

We have

$$
\begin{aligned}
e(Tz, w) &= \lim_{n\to\infty} e(Tz, y_{n+1}) \\
&= \lim_{n\to\infty} e(Tz, TSy_n) \\
&= \lim_{n\to\infty} c_1.\max\{d(z, Sy_n), e(y_n, Tz), e(y_n, TSy_n), d(z, STz)\} \\
&= \lim_{n\to\infty} c_1.\max\{d(z, x_n), e(y_n, Tz), e(y_n, y_{n+1}), d(z, STz)\} \\
&= c_1.\max\{d(z, z), e(w, Tz), e(w, w), d(z, STz)\} \\
&\le c_1.d(z, STz)
\end{aligned}
$$

Now

$$
\begin{aligned}
d(z, STz) &= \lim_{n\to\infty} d(x_n, STz) \\
&= \lim_{n\to\infty} d(Sy_n, STz) \\
&\le \lim_{n\to\infty} c_2.\max\{e(y_n, Tz), d(z, Sy_n), d(z, STz), e(Tz, TSy_n)\} \\
&= \lim_{n\to\infty} c_2.\max\{e(y_n, Tz), d(z, x_n), d(z, STz), e(Tz, y_{n+1})\} \\
&= c_2.\max\{e(w, Tz), d(z, z), d(z, STz), e(Tz, w)\} \\
&\le c_2.e(Tz, w)
\end{aligned}
$$

Hence

$e(Tz, w) \le c_1c_2.e(Tz, w) < e(Tz, w)$ (since $c_1c_2 < 1$) which is a contradiction.

Thus Tz = w.

Now we prove Sw = z.

Suppose $Sw \neq z$.

Then we have

$$
\begin{aligned}
d(Sw, z) &= \lim_{n\to\infty} d(Sw, x_{n+1}) \\
&= \lim_{n\to\infty} d(Sw, STx_n) \\
&\le \lim_{n\to\infty} c_2.\max\{e(w, Tx_n), d(x_n, Sw), d(x_n, STx_n), e(Tx_n, TSw)\} \\
&= \lim_{n\to\infty} c_2.\max\{e(w, y_{n+1}), d((x_n, Sw), d(x_n, x_{n+1}), e(y_{n+1}, TSw)\} \\
&\le c_2.e(w, TSw)
\end{aligned}
$$

Now

$$\begin{aligned} e(w, TSw) &= \lim_{n\to\infty} e(y_{n+1}, TSw) \\ &= \lim_{n\to\infty} e(Tx_n, TSw) \\ &\le \lim_{n\to\infty} c_1.\max\{d(x_n, Sw), e(w, Tx_n), e(w, TSw), d(x_n,STx_n)\} \\ &\le c_1.d(Sw,z) \end{aligned}$$

Hence

$$d(Sw, z) \le c_1c_2.d(Sw, z) < d(Sw, z) \ (\because c_1c_2 < 1), \text{ which is a contradiction.}$$

Thus $Sw = z$.

We have $STz = Sw = z$ and $TSw = Tz = w$. Thus the point z is a fixed point of ST in X and the point w is a fixed point of TS in Y.

Uniqueness: Let $z' \neq z$ in X be another fixed point of ST in X.

We have

$$\begin{aligned} d(z, z') &= d(Sw, STz') \\ &\le c_2.\max\{e(w, Tz'), d(z', Sw), d(z', STz'), e(Tz',w)\} \\ &\le c_2.e(Tz',w) \end{aligned}$$

Now

$$\begin{aligned} e(Tz',w) &= e(Tz', y_{n+1}) \\ &= e(Tz', TSy_n) \\ &\le c_1.\max\{d(z', Sy_n), e(y_n, Tz'), e(y_n, TSy_n), d(z', STz')\} \\ &\le c_1.d(z', z) \end{aligned}$$

Hence

$$d(z, z') \le c_1c_2.d(z, z') < d(z, z') \quad (\text{since } c_1c_2 < 1), \text{ which is a contradiction.}$$

So the point z is a unique fixed point of ST. Similarly, we prove the point w is also a unique point of TS. This completes the proof

Remark 2.8: If (X, d) and (Y, e) are the same metric spaces, then by the above theorem 2.7,we get the following theorem as corollary.

Corollary2.9: Let (X, d) be a complete metric space. If S and T are mappings from X into itself satisfying the following conditions:

$$d(Tx, TSy) \le c_1.\max\{d(x, Sy), d(y, Tx), d(y, TSy), d(x, STx)\}$$
$$d(Sy, STx) \le c_2.\max\{d(y, Tx), d(x, Sy), d(x, STx), d(Tx, TSy)\}$$

for all x, y in X where $0\le c_1, c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in X. Further, $Tz = w$ and $Sw = z$ and if $z = w$, then z is the unique common fixed point of S and T.

Theorem 2.10: Let (X, d) and (Y, e) be complete metric spaces. If T is a mapping from X into Y and S is a mapping from Y into X satisfying the following conditions

$$e(Tx, TSy) \le c_1.\max\{d(x, Sy), e(y, Tx), e(y, TSy), d(x, STx), d(Sy,STx)\} \qquad (1)$$
$$d(Sy, STx) \le c_2.\max\{e(y, Tx), d(x, Sy), d(x, STx), e(Tx,TSy), e(y,TSy)\} \qquad (2)$$

for all x in X and y in Y where $0\le c_1 < 1$ and $0 \le c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further $Tz =w$ and $Sw = z$.

Proof: Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X and a sequence $\{y_n\}$ in Y, as follows:

$$x_n = (ST)^n x_0, \ y_n = T(x_{n-1}) \ \text{ for } n = 1,2,\ldots .$$

We have

$$\begin{aligned} d(x_n, x_{n+1}) &= d((ST)^n x_0 , (ST)^{n+1} x_0)) \\ &= d(S(T(ST)^{n-1} x_0 , ST(ST)^n x_0) \\ &= d(ST(x_{n-1}), STx_n) \\ &= d(Sy_n, STx_n) \\ &\leq c_2.\max\{e(y_n,Tx_n), d(x_n,Sy_n), d(x_n,STx_n), e(Tx_n,TSy_n), e(y_n, TSy_n)\} \\ &= c_2.\max\{e(y_n, y_{n+1}), d(x_n, x_n), d(x_n, x_{n+1}), e(y_{n+1}, y_{n+1}), e(y_n, y_{n+1})\} \\ &= c_2. \max\{ e(y_n, y_{n+1}), 0, d(x_n, x_{n+1}), 0, e(y_n, y_{n+1})\} \\ &\leq c_2. e(y_n, y_{n+1}) \end{aligned}$$

Now

$$\begin{aligned} e(y_n, y_{n+1}) &= e(Tx_{n-1}, Tx_n) \\ &= e(Tx_{n-1}, TSy_n) \\ &\leq c_1. \max\{d(x_{n-1}, Sy_n), e(y_n, Tx_{n-1}) , e(y_n, TSy_n), d(x_{n-1}, STx_{n-1}), d(Sy_n, STx_{n-1})\} \\ &= c_1. \max\{d(x_{n-1}, x_n), e(y_n, y_n), e(y_n, y_{n+1}), d(x_{n-1,} x_n), d(x_{n-1}, x_n\}\} \\ &\leq c_1. d(x_{n-1} , x_n) \end{aligned}$$

Hence

$$\begin{aligned} d(x_n ,x_{n+1}) &\leq c_1c_2.d(x_{n-1}, x_n) \\ &\vdots \\ &\leq (c_1c_2)^n d(x_0,x_1) \to 0 \text{ as } n\to\infty \quad (\text{since } 0 \leq c_1c_2 <1) \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, it converges to a point z in X. Similarly, we can prove that the sequence $\{y_n\}$ is also a Cauchy sequence in (Y, e). Since (Y,e) is complete, it converges to a point w in Y.

Now we prove Tz = w.

Suppose $Tz \neq w$

We have

$$\begin{aligned} e(Tz, w) &= \lim_{n\to\infty} e(Tz, y_{n+1}) \\ &= \lim_{n\to\infty} e(Tz, TSy_n) \\ &\leq \lim_{n\to\infty} c_1.\max\{d(z, Sy_n), e(y_n,Tz),e(y_n,TSy_n), d(z, STz), d(Sy_{n,} STz)\} \\ &= \lim_{n\to\infty} c_1.\max \{d(z, x_n), e(y_n, Tz), e(y_n, y_{n+1}), d(z, STz), d(x_n, STz)\} \\ &\leq c_1.d(z, STz) \end{aligned}$$

Now

$$\begin{aligned} d(z, STz) &= \lim_{n\to\infty} d(x_n, STz) \\ &= \lim_{n\to\infty} d(Sy_n, STz) \\ &\leq \lim_{n\to\infty} c_2.\max\{e(y_n,Tz), d(z, Sy_n), d(z, STz), e(Tz, TSy_n), e(y_n, TSy_n)\} \\ &= \lim_{n\to\infty} c_2.\max\{e(y_n,Tz), d(z, x_n), d(z, STz), e(Tz, y_{n+1}), e(y_n, y_{n+1})\} \\ &\leq c_2.e(Tz, w) \end{aligned}$$

Hence

$$e(Tz, w) \leq c_1c_2.e(Tz, w) < e(Tz, w) \quad (\text{since } c_1c_2 < 1), \text{ which is a contradiction.}$$

Thus Tz = w.

To prove that Sw = z.

Suppose that $Sw \neq z$.

$$d(Sw, z) = \lim_{n\to\infty} d(Sw, x_{n+1})$$

$$= \lim_{n\to\infty} d(Sw, STx_n)$$
$$\leq \lim_{n\to\infty} c_2.\max\{e(w, Tx_n), d(x_n, Sw), d(x_n, STx_n), e(Tx_n, TSw), e(y_n, TSx_n)\}$$
$$= \lim_{n\to\infty} c_2.\max\{e(w, y_{n+1}), d(x_n, Sw), d(x_n, x_{n+1}), e(y_{n+1}, TSw), e(y_n, y_{n+1})\}$$
$$\leq c_2.e(w, TSw)$$

Now

$$e(w, TSw) = \lim_{n\to\infty} e(y_{n+1}, TSw)$$
$$= \lim_{n\to\infty} e(Tx_n, TSw)$$
$$\leq \lim_{n\to\infty} c_1.\max\{d(x_n, Sw), e(w, Tx_n), e(w, TSw), e(Tx_n, TSw), e(y_n, TSx_n)\}$$
$$= \lim_{n\to\infty} c_1.\max\{d(x_n, Sw), e(w, y_{n+1}), e(w, TSw), d(x_n, x_{n+1}), d(Sw, x_{n+1})\}$$
$$\leq c_1.d(Sw, z)$$

Hence

$$d(Sw, z) \leq c_1c_2.d(Sw, z) < d(Sw, z) \text{ (since } c_1c_2 < 1\text{), which is a contradiction.}$$

Thus Sw = z.

We have STz = Sw = z and TSw = Tz = w. Thus the point z is a fixed point of ST in X and the point w is a fixed point of TS in Y.

Uniqueness: Let $z' \neq z$ be the another fixed point of ST in X.

We have

$$d(z, z') = d(Sw, STz')$$
$$\leq c_2.\max\{e(w,Tz'), d(z',Sw), d(z',STz'), e(Tz',TSw), e(w, TSw)\}$$
$$= c_2.\max\{e(w,Tz'), d(z',z), d(z',z'), e(Tz', w), e(w, w)\}$$
$$\leq c_2.e(Tz',w)$$

Now

$$e(Tz', w) = e(Tz', TSw)$$
$$\leq c_1.\max\{d(z', Sw), e(w, Tz'), e(z', TSz'), e(Tx_n, TSw), e(y_n, TSx_n)\}$$
$$= c_1.\max\{d(z', z), e(w, Tz'), e(z', TSz'), d(z', z), d(z, STz')$$
$$\leq c_1.d(z, z')$$

Hence

$$d(z, z') \leq c_1c_2.d(z, z') < d(z, z') \text{ (since } c_1c_2 < 1\text{), which is a contradiction.}$$

Thus $z = z'$.

So the point z is a unique fixed point z of ST. Similarly, we prove the point w is also a unique point of TS. This completes the proof.

Remark 2.11: If (X, d) and (Y, e) are the same metric spaces, then by the above theorem 2.10., we get the following theorem as corollary.

Corollary2.12: Let (X, d) be a complete metric space. If S and T are mappings from X into itself satisfying the following conditions:

$$d(Tx, TSy) \leq c_1.\max\{d(x, Sy), d(y, Tx), d(y, TSy), d(x, STx)\}$$
$$d(Sy, STx) \leq c_2.\max\{d(y, Tx), d(x, Sy), d(x, STx), d(Tx, TSy)\}$$

for all x, y in X where $0 \leq c_1, c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in X. Further, Tz = w and Sw = z and if z = w, then z is the unique common fixed point of S and T.

Theorem2.13: Let (X, d) and (Y, e) be complete metric spaces. If T is a mapping from X into Y and S is a mapping from Y into X satisfying the following conditions

$$e(Tx, TSy) \leq c_1.\max\{d(x, Sy), d(Sy, STx), e(y, Tx) + e(y, TSy), d(x, STx)\} \quad (1)$$
$$d(Sy, STx) \leq c_2.\max\{d(x, Sy) + d(x, STx), e(y, TSy), e(y, Tx), e(Tx,TSy)\} \quad (2)$$

for all x in X and y in Y where $0 \le c_1 < 1$ and $0 \le c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further Tz =w and Sw = z.

Proof: Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X and a sequence $\{y_n\}$ in Y as follows:

$$x_n = (ST)^n x_0, \quad y_n = T(x_{n-1}) \text{ for } n = 1,2,\dots .$$

We have

$$\begin{aligned} d(x_n, x_{n+1}) &= d((ST)^n x_0, (ST)^{n+1} x_0)) \\ &= d(S(T(ST)^{n-1} x_0, ST(ST)^n x_0) \\ &= d(ST(x_{n-1}), STx_n) \\ &= d((Sy_n, STx_n) \\ &\le c_2.\max\{d(x_n, Sy_n) + d(x_n, STx_n), e(y_n, TSy_n), e(y_n, Tx_n), e(Tx_n, TSy_n)\} \\ &= c_2.\max\{d(x_n, x_n) + d(x_n, x_{n+1}), e(y_n, y_{n+1}), e(y_n, y_{n+1}), e(y_{n+1}, y_{n+1})\} \\ &\le c_2.e(y_n, y_{n+1}) \end{aligned}$$

Now

$$\begin{aligned} e(y_n, y_{n+1}) &= e(Tx_{n-1}, Tx_n) \\ &= e(Tx_{n-1}, TSy_n) \\ &\le c_1.\max\{d(x_{n-1}, Sy_n), d(Sy_n, STx_{n-1}), e(y_n, Tx_{n-1}) + e(y_n, TSy_n), d(x_{n-1}, STx_{n-1})\} \\ &= c_1.\max\{d(x_{n-1}, x_n), d(x_n, x_n) + e(y_n, y_n), e(y_n, y_{n+1}), d(x_{n-1}, x_n)\} \\ &= c_1.\max\{d(x_{n-1}, x_n), 0, e(y_n, y_{n+1}), d(x_{n-1}, x_n)\} \\ &\le c_1. d(x_{n-1}, x_n) \end{aligned}$$

Hence

$$\begin{aligned} d(x_n, x_{n+1}) &\le c_1c_2.d(x_{n-1}, x_n) \\ &\vdots \\ &\le (c_1c_2)^n d(x_0,x_1) \to 0 \text{ as } n\to\infty \quad (\text{since } 0 \le c_1c_2 < 1) \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is complete, it converges to a point z in X. Similarly, we can prove that the sequence $\{y_n\}$ is also a Cauchy sequence in (Y,e). Since (Y, e) is complete, it converges to a point w in Y.

Now we prove Tz = w.

Suppose $Tz \ne w$.

We have

$$\begin{aligned} e(Tz, w) &= \lim_{n\to\infty} e(Tz, y_{n+1}) \\ &= \lim_{n\to\infty} e(Tz, TSy_n) \\ &\le \lim_{n\to\infty} c_1.\max\{d(z, Sy_n), d(Sy_n, STz), e(y_n, Tz), e(y_n, TSy_n), d(z, STz)\} \\ &= \lim_{n\to\infty} c_1.\max\{d(z, x_n), d(x_n, STz), e(y_n, Tz) + e(y_n, y_{n+1}), d(z, STz)\} \\ &\le c_1.d(z, STz) \end{aligned}$$

Now

$$\begin{aligned} d(z, STz) &= \lim_{n\to\infty} d(x_n, STz) \\ &= \lim_{n\to\infty} d(Sy_n, STz) \\ &\le \lim_{n\to\infty} c_2.\max\{d(z, Sy_n) + d(z, STz), e(y_{n.,}TSy_n), e(y_n, Tz), e(Tz, TSy_n)\} \\ &= \lim_{n\to\infty} c_2.\max\{d(z, x_n) + d(z, STz), e(y_n, y_{n+1}), e(y_n, Tz), e(Tz, y_{n+1})\} \\ &\le c_2.e(Tz, w) \end{aligned}$$

Hence

$e(Tz, w) \le c_1c_2.e(Tz, w) < e(Tz, w)$ (since $c_1c_2<1$) which is a contradiction.

Thus Tz = w.

Now we prove Sw = z.

Suppose Sw $\neq$ z.

We have

$$
\begin{aligned}
d(Sw, z) &= \lim_{n\to\infty} d(Sw, x_{n+1}) \\
&= \lim_{n\to\infty} d(Sw, STx_n) \\
&\le \lim_{n\to\infty} c_2.\max\{d(x_n, Sw) + d(x_n, STx_n), e(w, TSw), e(w, Tx_n), e(Tx_n, TSw)\} \\
&= \lim_{n\to\infty} c_2.\max\{d(x_n, Sw) + d(x_n, x_{n+1}), e(w, w), e(w, y_{n+1}), e(y_{n+1}, w)\} \\
&\le c_2.e(w, TSw)
\end{aligned}
$$

Now

$$
\begin{aligned}
e(w, TSw) &= \lim_{n\to\infty} e(y_{n+1}, TSw) \\
&= \lim_{n\to\infty} e(Tx_n, TSw) \\
&\le \lim_{n\to\infty} c_1.\max\{d(x_n, Sw), d(Sw, STx_n), e(w, Tx_n) + e(w, TSw), d(x_n, Tx_n)\} \\
&= \lim_{n\to\infty} c_1.\max\{d(x_n, Sw), d(Sw, x_{n+1}), e(w, y_{n+1}) + e(w, TSw), d(x_n, y_{n+1})\} \\
&\le c_1.d(z, Sw)
\end{aligned}
$$

Hence

$d(Sw, z) \le c_1c_2.d(z, Sw) < d(Sw, z)$ (since $c_1c_2 < 1$) which is a contradiction.

Thus Sw = z.

We have STz = Sw = z and TSw = Tz = w. Thus the point z is a fixed point of ST in X and the point w is a fixed point of TS in Y.

Uniqueness: Let $z' \neq z$ be the another fixed point of ST in X.

We have

$$
\begin{aligned}
d(z, z') &= d(Sw, STz') \\
&\le c_2.\max\{d(z', Sw) + d(z', STz'), e(w, TSw), e(w, Tz'), e(Tz', TSw)\} \\
&= c_2.\max\{d(z', z) + d(z', z'), e(w, w), e(w, Tz'), e(Tz', w)\} \\
&\le c_2.e(w, Tz')
\end{aligned}
$$

Now

$$
\begin{aligned}
e(Tz', w) &= e(Tz', TSw) \\
&\le c_1.\max\{d(z', Sw), d(Sw, STz'), e(w, Tz') + e(w, TSw)\ d(z', STz')\} \\
&= c_1.\max\{d(z', z), d(z, z'), e(w, Tz') + e(w, w), d(z', z')\} \\
&\le c_1.\ d(z, z')
\end{aligned}
$$

Hence

$d(z, z') \le c_1c_2.d(z, z') < d(z, z')$ which is a contradiction.

Thus $z = z'$.

So the point z is a unique fixed point of ST. Similarly, we prove the point w is also a unique point of TS. This completes the proof.

Remark 2.14: If (X, d) and (Y, e) are the same metric spaces, then by the above theorem 2.13, we get the following theorem, as corollary.

Corollary2.15: Let (X, d) be a complete metric space. If S and T are mappings from X into itself satisfying the following conditions:

$d(Tx, TSy) \leq c_1.\max\{d(x, Sy), d(Sy, STx), d(y, Tx) + d(y, TSy), d(x, STx)\}$
$d(Sy, STx) \leq c_2.\max\{d(x, Sy) + d(x, STx), d(y, TSy), d(y, Tx), d(Tx,TSy)\}$

for all x, y in X where $0 \leq c_1, c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in X. Further, Tz = w and Sw = z and if z = w, then z is the unique common fixed point of S and T.

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