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CLASSES $\Lambda^{m,n}(\psi, p, r)$ AND DOUBLE TRIGONOMETRIC FOURIER SERIES

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ABSTRACT.

In this paper we generalize the Lipschitz classes $\Lambda^{m,n}(\alpha, \beta, p, r)$ defined in [6]. Here separately are given, necessary and sufficient conditions, so that a function with double Fourier series belongs the generalized classes $\Lambda^{m,n}(\psi, p, r)$.

Key words: Lipschitz classes, double trigonometric Fourier series.

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1. INTRODUCTION:

Let $f(x, y) \in L(\mathbf{T}^2)$, $\mathbf{T}^2 \equiv [-\pi, \pi] \times [-\pi, \pi]$, be $2\pi-$periodic function with respect to each variable and even with respect to both variables x, y; even with respect to x and odd with respect to y; odd with respect to x and even with respect to y; odd with respect to both variables x, y, with its Fourier series

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \lambda_{i,k} a_{i,k} \cos ix \cos ky, \tag{1.1}$$

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \lambda_{i,k} a_{i,k} \cos ix \sin ky, \tag{1.2}$$

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \lambda_{i,k} a_{i,k} \sin ix \cos ky, \tag{1.3}$$

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{i,k} \sin ix \sin ky, \tag{1.4}$$

respectively, where

$$\lambda_{i,k} = \begin{cases} \frac{1}{4} & \text{if } i = k = 0 \\ \frac{1}{2} & \text{if } i = 0,\ k > 0;\ i > 0,\ k = 0 \\ 1 & \text{if } i,\ k > 0. \end{cases}$$

We say that $f \in AW^p, (1 < p < \infty)$ if

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} (ik)^{p-2} \left(\sum_{j=i}^{\infty}\sum_{s=k}^{\infty} \left| \Delta_{1,1} a_{j,s} \right| \right)^p < \infty,$$

where $\Delta_{1,1} a_{j,s} = a_{j,s} - a_{j+1,s} - a_{j,s+1} + a_{j+1,s+1}$ (see [2]).

Is said to be $f \in \Lambda^{m,n}(\alpha, \beta, p, r)$ if

$$\|f\|_{\alpha,\beta,p,r}^{(m,n)} \equiv \left\{ \int_0^1 \int_0^1 \left[\int_0^{2\pi} \int_0^{2\pi} \frac{|\Delta_{m,n} f(x, y; t, \tau)|^p}{t^{\alpha p} \tau^{\beta p}} dx dy \right]^{\frac{r}{p}} \frac{dt d\tau}{t\tau} \right\}^{\frac{1}{r}} < \infty,$$

where $1 < p < \infty, 1 \le r < \infty, \alpha, \beta > 0, m, n \in \mathbf{N}$ and

$$\Delta_{m,n} f(x, y; t, \tau) = \sum_{i=0}^{m}\sum_{k=0}^{n} (-1)^{i+k} C_m^i C_n^k f\left[x + (m-2i)t, y + (n-2k)\tau\right].$$

As usually the expression $(h, \eta > 0)$

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$$\Omega_p^{(m,n)}(h,\eta;f)=\sup_{|t|\le h}\sup_{|\tau|\le\eta}\left\{\int_0^{2\pi}\int_0^{2\pi}\left|\Delta_{m,n}f(x,y;t,\tau)\right|^p dxdy\right\}^{\frac{1}{p}}$$

is the integral modulus of smoothness of order (m,n).

Let $\alpha_i(t_i)$, $(i=1,2)$ be positive functions that satisfy the following properties:

(1) The functions $\alpha_i(t_i)$ are measurable on $[0,2\pi]$ and

(2) Integrable on $[\delta_i, 2\pi]$ for each $\delta_i\in(0,2\pi)$.

A function $\psi(t_1,t_2)=\alpha_1(t_1)\alpha_2(t_2)$ satisfies $\sigma=(\sigma_1,\sigma_2)$-condition (see [2]) if there exist real numbers $\delta_i\in(0,2\pi)$ and $\grave{o}_i>0$, $(i=1,2)$ so that:

$$\int_0^{\delta_i}\alpha_i(t_i)t_i^{\sigma_i}dt_i<\infty \qquad \text{and} \qquad \int_0^{\delta_i}\alpha_i(t_i)t_i^{\sigma_i-\grave{o}_i}dt_i=\infty.$$

We say that $f\in\Lambda^{m,n}(\psi,p,r)$ if

$$\|f\|_{\psi,p,r}^{(m,n)}\equiv\left\{\int_0^1\int_0^1\psi(t,\tau)\left[\int_0^{2\pi}\int_0^{2\pi}|\Delta_{m,n}f(x,y;t,\tau)|^p\, dxdy\right]^{\frac{r}{p}}dtd\tau\right\}^{\frac{1}{r}}<\infty,$$

where the function $\psi(t,\tau)$ satisfies $\sigma=(\sigma_1,\sigma_2)$-condition.

We notice that if we give the values $\alpha_1(t)=t^{-\alpha r-1}$ and $\alpha_2(\tau)=\tau^{-\beta r-1}$, $(\alpha,\beta>0)$ to the function $\psi(t,\tau)$, we get the classes $\Lambda^{m,n}(\alpha,\beta,p,r)$ defined in [6]. So, the classes $\Lambda^{m,n}(\psi,p,r)$ are generalization of the classes $\Lambda^{m,n}(\alpha,\beta,p,r)$.

The aim of this paper is to study, apart, necessary conditions and sufficient conditions which guarantee that a function belongs the classes $\Lambda^{m,n}(\psi,p,r)$. We recall that for one dimensional case these conditions are given by present author in [7]. We make mention that for the classes $\Lambda^{m,n}(\alpha,\beta,p,r)$ these condition are studied in [6]. There are given, as well, at the same time necessary and sufficient conditions so that a function with double Fourier series belongs to the same classes. Due to chronology it is important to point out that some conditions for more simply classes, than those that Tevzadze considered, are given by R. Askey and S. Waigner [5], and M. Izumi and S. Izumi [3], as well.

2. SEVERAL LEMMAS:

In this section we will give some auxiliary statements which we need to prove the main results. Below, some lemmas in two-dimensional case are proved for what we believe that they can be useful to another applications concerning to the double numerical series with positive terms. We start with the following lemma proved in [6].

Lemma: 2.1. Let $f\in AW^p$, $(1<p<\infty)$. Then

$$\{\Omega_p^{(m,n)}(h,\eta,f)\}^p\le A_{m,n,p}\Big\{h^{mp}\eta^{np}\sum_{i\le\left[\frac{1}{h}\right]}\sum_{k\le\left[\frac{1}{\eta}\right]}i^{(m+1)p-2}k^{(n+1)p-2}\left(\sum_{j=i}^{\infty}\sum_{s=k}^{\infty}\left|\Delta_{1,1}a_{j,s}\right|\right)^p$$

$$+h^{mp}\sum_{i\le\left[\frac{1}{h}\right]}\sum_{k>\left[\frac{1}{\eta}\right]}i^{(m+1)p-2}k^{p-2}\left(\sum_{j=i}^{\infty}\sum_{s=k}^{\infty}\left|\Delta_{1,1}a_{j,s}\right|\right)^p$$

$$+\eta^{np}\sum_{i>\left[\frac{1}{h}\right]}\sum_{k\le\left[\frac{1}{\eta}\right]}i^{p-2}k^{(n+1)p-2}\left(\sum_{j=i}^{\infty}\sum_{s=k}^{\infty}\left|\Delta_{1,1}a_{j,s}\right|\right)^p$$

$$+\sum_{i>\left[\frac{1}{h}\right]}\sum_{k>\left[\frac{1}{\eta}\right]}(ik)^{p-2}\left(\sum_{j=i}^{\infty}\sum_{s=k}^{\infty}\left|\Delta_{1,1}a_{j,s}\right|\right)^p\Big\}.$$

Lemma: 2.2. [4] Let a_ν, b_ν and β_n be numbers such that $a_\nu\ge0$, $b_\nu\ge0$ and $\sum_{\nu=n}^{\infty}a_\nu=a_n\beta_n,(n=1,2,\ldots)$. Then:

(1) For $0<p\le1$ we have

$$\sum_{\nu=1}^{\infty}a_\nu\left(\sum_{\mu=1}^{\nu}b_\mu\right)^p\ge p^p\sum_{\nu=1}^{\infty}a_\nu\left(b_\nu\beta_\nu\right)^p;$$

(2) For $1\le p<\infty$ we have

$$\sum_{\nu=1}^{\infty}a_\nu\left(\sum_{\mu=1}^{\nu}b_\mu\right)^p\le p^p\sum_{\nu=1}^{\infty}a_\nu\left(b_\nu\beta_\nu\right)^p.$$

Lemma: 2.3. [4] Let a_ν, b_ν and γ_n be numbers such that $a_\nu\ge0$, $b_\nu\ge0$ and $\sum_{\nu=1}^{n}a_\nu=a_n\gamma_n$, $(n=1,2,\ldots)$. Then:

(1) For $0<p\le1$, we have

$$\sum_{\nu=1}^{\infty}a_\nu\left(\sum_{\mu=\nu}^{\infty}b_\mu\right)^p\ge p^p\sum_{\nu=1}^{\infty}a_\nu\left(b_\nu\gamma_\nu\right)^p;$$

(2) For $1\le p<\infty$, we have

$$\sum_{\nu=1}^{\infty} a_\nu \left(\sum_{\mu=\nu}^{\infty} b_\mu\right)^p \leq p^p \sum_{\nu=1}^{\infty} a_\nu \left(b_\nu \gamma_\nu\right)^p.$$

Now, because of their simplicity, we will prove briefly four lemmas for which we spoke at the beginning of this section.

Lemma: 2.4. Let $a_{\mu,\nu}$, $b_{\mu,\nu}$, $\beta^{(1)}_{m,n}$ and $\beta^{(2)}_{m,n}$ be numbers such that $a_{\mu,\nu} \geq 0$, $b_{\mu,\nu} \geq 0$, $\sum_{\mu=m}^{\infty} a_{\mu,n}\left(\beta^{(2)}_{\mu,n}\right)^p = \beta^{(1)}_{m,n} a_{m,n}\left(\beta^{(2)}_{m,n}\right)^p$, and $\sum_{\nu=n}^{\infty} a_{m,\nu} = a_{m,n}\beta^{(2)}_{m,n}, (m,n=1,2,\ldots)$. Then:

(1) For $0 < p \leq 1$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\left(\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} b_{\mu,\nu}\right)^p \geq p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\left(b_{m,n}\beta^{(1)}_{m,n}\beta^{(2)}_{m,n}\right)^p;$$

(2) For $1 \leq p < \infty$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\left(\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} b_{\mu,\nu}\right)^p \leq p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\left(b_{m,n}\beta^{(1)}_{m,n}\beta^{(2)}_{m,n}\right)^p.$$

Proof. Putting $\sum_{\mu=1}^{m} b_{\mu,\nu} = c_{m,\nu}$ and using twice Lemma 2.2 for $0 < p \leq 1$ we have:

$$\begin{aligned}
\sum_{m=1}^{M}\sum_{n=1}^{N} a_{m,n}\left(\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} b_{\mu,\nu}\right)^p &= \sum_{m=1}^{M}\left[\sum_{n=1}^{N} a_{m,n}\left(\sum_{\nu=1}^{n} c_{m,\nu}\right)^p\right] \\
&\geq p^p \sum_{m=1}^{M}\sum_{n=1}^{N} a_{m,n}\left(c_{m,n}\beta^{(2)}_{m,n}\right)^p \\
&= p^p \sum_{n=1}^{N}\left[\sum_{m=1}^{M} a_{m,n}\left(\beta^{(2)}_{m,n}\right)^p\left(\sum_{\mu=1}^{m} b_{\mu,n}\right)^p\right] \\
&\geq p^{2p}\sum_{m=1}^{M}\sum_{n=1}^{N} a_{m,n}\left(\beta^{(2)}_{m,n}\right)^p\left(b_{m,n}\beta^{(1)}_{m,n}\right)^p \\
&= p^{2p}\sum_{m=1}^{M}\sum_{n=1}^{N} a_{m,n}\left(b_{m,n}\beta^{(1)}_{m,n}\beta^{(2)}_{m,n}\right)^p.
\end{aligned}$$

Passing on limit when $M, N \to \infty$ to the last inequality we get required inequality.

2. The Lemma 2.4 for $1 \leq p < \infty$ can be prove in the same way.

Lemma: 2.5. Let $a_{\mu,\nu}$, $b_{\mu,\nu}$, $\gamma^{(1)}_{m,n}$ and $\gamma^{(2)}_{m,n}$ be numbers such that $a_{\mu,\nu} \geq 0$, $b_{\mu,\nu} \geq 0$, $\sum_{\mu=1}^{m} a_{\mu,n} = a_{m,n}\gamma^{(1)}_{m,n}$, and $\sum_{\nu=1}^{n} a_{m,\nu}\left(\gamma^{(1)}_{m,\nu}\right)^p = \gamma^{(2)}_{m,n} a_{m,n}\left(\gamma^{(1)}_{m,n}\right)^p$, $(m,n=1,2,\ldots)$. Then:

(1) For $0 < p \leq 1$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\left(\sum_{\mu=m}^{\infty}\sum_{\nu=n}^{\infty} b_{\mu,\nu}\right)^p \geq p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\left(b_{m,n}\gamma^{(1)}_{m,n}\gamma^{(2)}_{m,n}\right)^p;$$

(2) For $1 \leq \mathrm{p} < \infty$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\left(\sum_{\mu=m}^{\infty}\sum_{\nu=n}^{\infty} b_{\mu,\nu}\right)^p \leq p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\left(b_{m,n}\gamma^{(1)}_{m,n}\gamma^{(2)}_{m,n}\right)^p.$$

Proof: The proof of this Lemma is very similiar to the proof of the Lemma 2.4. In this case it is enough to apply twice Lemma 2.3; therefore we leave it to the reader.

Lemma: 2.6. Let $a_{\mu,\nu}$, $b_{\mu,\nu}$, $\beta^{(1)}_{m,n}$ and $\gamma^{(2)}_{m,n}$ be numbers such that $a_{\mu,\nu} \geq 0$, $b_{\mu,\nu} \geq 0$, $\sum_{\mu=m}^{\infty} a_{\mu,n} = a_{m,n}\beta^{(1)}_{m,n}$, and $\sum_{\nu=1}^{n} a_{m,\nu}\left(\beta^{(1)}_{m,\nu}\right)^p = \gamma^{(2)}_{m,n} a_{m,n}\left(\beta^{(1)}_{m,n}\right)^p$, $(m,n=1,2,\ldots)$. Then:

(1) For $0 < p \leq 1$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\left(\sum_{\mu=1}^{m}\sum_{\nu=n}^{\infty} b_{\mu,\nu}\right)^p \geq p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\left(b_{m,n}\beta^{(1)}_{m,n}\gamma^{(2)}_{m,n}\right)^p;$$

(2) For $1 \leq p < \infty$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\left(\sum_{\mu=1}^{m}\sum_{\nu=n}^{\infty} b_{\mu,\nu}\right)^p \leq p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\left(b_{m,n}\beta^{(1)}_{m,n}\gamma^{(2)}_{m,n}\right)^p.$$

Proof: Let us prove the case $1 \leq p < \infty$ (the case $0 < p \leq 1$ can be prove in a similar way). If we set $\sum_{\nu=n}^{\infty} b_{\mu,\nu} = c_{\mu,n}$ and applying the Lemma 2.2 we get

$$\sum_{m=1}^{M}\sum_{n=1}^{N}a_{m,n}\left(\sum_{\mu=1}^{m}\sum_{\nu=1}^{\infty}b_{\mu,\nu}\right)^p=\sum_{n=1}^{N}\left[\sum_{m=1}^{M}a_{m,n}\left(\sum_{\mu=1}^{m}c_{\mu,n}\right)^p\right]$$
$$\le p^p\sum_{n=1}^{N}\left[\sum_{m=1}^{M}a_{m,n}\left(c_{m,n}\beta^{(1)}_{m,n}\right)^p\right]$$
$$=p^p\sum_{m=1}^{M}\left[\sum_{n=1}^{N}a_{m,n}\left(\beta^{(1)}_{m,n}\right)^p\left(\sum_{\nu=n}^{\infty}b_{m,\nu}\right)^p\right].$$

Applying Lemma 2.3 to the last inequality we have

$$\sum_{m=1}^{M}\sum_{n=1}^{N}a_{m,n}\left(\sum_{\mu=1}^{m}\sum_{\nu=1}^{\infty}b_{\mu,\nu}\right)^p\le p^{2p}\sum_{m=1}^{M}\left[\sum_{n=1}^{N}a_{m,n}\left(\beta^{(1)}_{m,n}\right)^p\left(b_{m,n}\gamma^{(2)}_{m,n}\right)^p\right]$$
$$=p^{2p}\sum_{m=1}^{M}\sum_{n=1}^{N}a_{m,n}\left(b_{m,n}\beta^{(1)}_{m,n}\gamma^{(2)}_{m,n}\right)^p.$$

Letting $M,N\to\infty$ to the last inequality then the proof of the lemma completes.

Lemma: 2.7. Let $a_{\mu,\nu}$, $b_{\mu,\nu}$, $\beta^{(2)}_{m,n}$ and $\gamma^{(1)}_{m,n}$ be numbers such that $a_{\mu,\nu}\ge 0$, $b_{\mu,\nu}\ge 0$, $\sum_{\nu=n}^{\infty}a_{m,\nu}=a_{m,n}\beta^{(2)}_{m,n}$, and

$$\sum_{\mu=1}^{m}a_{\mu,n}\left(\beta^{(2)}_{\mu,n}\right)^p=\gamma^{(1)}_{m,n}a_{m,n}\left(\beta^{(2)}_{m,n}\right)^p,\ (m,n=1,2,\ldots).$$

Then:

(1) For $0<p\le 1$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{m,n}\left(\sum_{\mu=m}^{\infty}\sum_{\nu=1}^{n}b_{\mu,\nu}\right)^p\ge p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{m,n}\left(b_{m,n}\beta^{(2)}_{m,n}\gamma^{(1)}_{m,n}\right)^p;$$

(2)For $1\le p<\infty$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{m,n}\left(\sum_{\mu=m}^{\infty}\sum_{\nu=1}^{n}b_{\mu,\nu}\right)^p\le p^{2p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{m,n}\left(b_{m,n}\beta^{(2)}_{m,n}\gamma^{(1)}_{m,n}\right)^p.$$

The proof of the Lemma 2.7 is similiar to the proof of the Lemma 2.6 therefore we omit it.

Lemma: 2.8. [9] Let $u_1\ge 0$, $u_2\ge 0$. Then for each $\kappa>0$ the following double inequality holds

$$C_1(\kappa)\left(u_1^\kappa+u_2^\kappa\right)\le\left(u_1+u_2\right)^\kappa\le C_2(\kappa)\left(u_1^\kappa+u_2^\kappa\right).$$

Lemma: 2.9. [9] Let μ, τ and a_ν be numbers such that $0<\mu<\tau<\infty$ and $a_\nu\ge 0$. Then

$$\left(\sum_{\nu=1}^{\infty}a_\nu^\tau\right)^{\frac{1}{\tau}}\le\left(\sum_{\nu=1}^{\infty}a_\nu^\mu\right)^{\frac{1}{\mu}}.$$

In what follows we denote by C a positive constant that depends only on on m,n,p,r and may be different in different relations.

1. 3. MAIN RESULTS:

For $\mu,\nu\in\mathbf{N}$, let us denote

$$A(\mu):=\int_{1/(\mu+1)}^{1/\mu}\alpha_1(t)dt$$
$$A(\nu):=\int_{1/(\nu+1)}^{1/\nu}\alpha_2(\tau)d\tau$$
$$b(\mu):=b_1(\mu)+b_2(\mu)=\mu^{mr}\int_0^{1/\mu}\alpha_1(t)t^{mr}dt+\int_{1/(\mu+1)}^{1}\alpha_1(t)dt$$
$$b(\nu):=b_1(\nu)+b_2(\nu)=\nu^{nr}\int_0^{1/\nu}\alpha_2(\tau)\tau^{nr}d\tau+\int_{1/(\nu+1)}^{1}\alpha_2(\tau)d\tau$$
$$B_{i,k}:=\sum_{j=i}^{\infty}\sum_{s=k}^{\infty}\left|\Delta_{1,1}a_{j,s}\right|;i,k=1,2,\ldots$$

and $\quad A(\mu,\nu):=A(\mu)A(\nu),\quad b(\mu,\nu):=b(\mu)b(\nu).$

Theorem: 3.1. Let m,n be natural numbers,

$f\in AW^p, 1<p<\infty$ and $1\le r<\infty$. If

$$\sum_{j=1}^{\infty}\sum_{s=1}^{\infty}\left|\Delta_{1,1}a_{j,s}\right|<\infty,$$

where $a_{j,s}$ are coefficients either of the series (1.1), (1.2), (1.3) or (1.4), then:

(1) For $p\le r$ we have

$$\left\|f\right\|^{(m,n)}_{\psi,p,r}\le C\left\{\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}b(\mu,\nu)(\mu\nu)^{r\left(1-\frac{2}{p}\right)}B^r_{\mu,\nu}\left[\frac{b(\mu,\nu)}{A(\mu,\nu)}\right]^{\frac{r}{p}-1}\right\}^{\frac{1}{r}};$$

(2) For $p>r$ we have

$$\left\|f\right\|^{(m,n)}_{\psi,p,r}\le C\left\{\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}b(\mu,\nu)(\mu\nu)^{r\left(1-\frac{2}{p}\right)}B^r_{\mu,\nu}\right\}^{1/r}.$$

Proof:. It is obvious that

$$\left\{\left\|f\right\|_{\psi,p,r}^{(m,n)}\right\}^r \le \int_0^1\int_0^1 \alpha_1(t)\alpha_2(\tau)\left[\Omega_p^{(m,n)}(t,\tau,f)\right]^r dtd\tau$$

$$=\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}\int_{1/(\mu+1)}^{1/\mu}\int_{1/(\nu+1)}^{1/\nu}\alpha_1(t)\alpha_2(\tau)\left[\Omega_p^{(m,n)}(t,\tau,f)\right]^r dtd\tau$$

$$=\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu)A(\nu)\left[\Omega_p^{(m,n)}\left(1/\mu,1/\nu;f\right)\right]^r.$$

Using Lemma 2.1 and applying Lemma 2.8 three times we have that

$$\left\{\left\|f\right\|_{\psi,p,r}^{(m,n)}\right\}^r \le C\Bigg\{\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu)A(\nu)\mu^{-mr}\nu^{-nr}\left[\sum_{i=1}^{\mu}\sum_{k=1}^{\nu}i^{(m+1)p-2}k^{(n+1)p-2}B_{i,k}^p\right]^{r/p}$$

$$+\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu)A(\nu)\mu^{-mr}\left[\sum_{i=1}^{\mu}\sum_{k=\nu}^{\infty}i^{(m+1)p-2}k^{p-2}B_{i,k}^p\right]^{r/p}$$

$$+\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu)A(\nu)\nu^{-nr}\left[\sum_{i=\mu}^{\infty}\sum_{k=1}^{\nu}i^{p-2}k^{(n+1)p-2}B_{i,k}^p\right]^{r/p}$$

$$+\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu)A(\nu)\left[\sum_{i=\mu}^{\infty}\sum_{k=\nu}^{\infty}i^{p-2}k^{p-2}B_{i,k}^p\right]^{r/p}\Bigg\}\equiv\sum_{j=1}^{4}P_j. \qquad 3.1$$

Let $r\ge p$. Let us consider first the quantity P_1. Then the Lemma 2.4 gives

$$P_1\le C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu)A(\nu)\mu^{-mr}\nu^{-nr}\left[\mu^{(m+1)p-2}\nu^{(n+1)p-2}B_{\mu,\nu}^p\beta_{\mu,\nu}^{(1)}\beta_{\mu,\nu}^{(2)}\right]^{r/p}$$

$$=C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu)A(\nu)(\mu\nu)^{r\left(1-\frac{2}{p}\right)}B_{\mu,\nu}^r\left[\beta_{\mu,\nu}^{(1)}\beta_{\mu,\nu}^{(2)}\right]^{r/p}.$$

It is obvious that $\beta_{\mu,\nu}^{(1)}$ depends only on μ and $\beta_{\mu,\nu}^{(2)}$ depends only on ν and for them the following relations hold:

$$\beta_\mu^{(1)}=\frac{\mu^{mr}}{A(\mu)}\sum_{i=\mu}^{\infty}A(i)i^{-mr}$$

$$=\frac{\mu^{mr}}{A(\mu)}\sum_{i=\mu}^{\infty}\left(\frac{i+1}{i}\right)^{mr}\cdot\frac{1}{(i+1)^{mr}}\int_{1/i}^{1/(i+1)}\alpha_1(t)dt$$

$$\le 2^{mr}\frac{\mu^{mr}}{A(\mu)}\int_0^{1/\mu}\alpha_1(t)t^{mr}dt$$

$$=2^{mr}\frac{b_1(\mu)}{A(\mu)}$$

and in the same manner we can show that $\beta_\nu^{(2)}\le 2^{nr}\dfrac{b_1(\nu)}{A(\nu)}$.

Therefore we have

$$P_1\le C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu,\nu)(\mu\nu)^{r\left(1-\frac{2}{p}\right)}B_{\mu,\nu}^r\left[\frac{b_1(\mu)b_1(\nu)}{A(\mu,\nu)}\right]^{\frac{r}{p}}. \qquad 3.2$$

Now we etimate P_2. By the Lemma 2.6 we get

$$P_2\le C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu)A(\nu)\mu^{-mr}\left[\mu^{(m+1)p-2}\nu^{p-2}B_{\mu,\nu}^p\beta_{\mu,\nu}^{(1)}\gamma_{\mu,\nu}^{(2)}\right]^{r/p}$$

$$=C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu)A(\nu)(\mu\nu)^{r\left(1-\frac{2}{p}\right)}B_{\mu,\nu}^r\left[\beta_{\mu,\nu}^{(1)}\gamma_{\mu,\nu}^{(2)}\right]^{r/p}.$$

As we showed $\beta_\mu^{(1)}\le 2^{mr}\dfrac{b_1(\mu)}{A(\mu)}$, and since

$$A(\nu)\gamma_\nu^{(2)}=\sum_{j=1}^{\nu}A(j)\Rightarrow\gamma_\nu^{(2)}=\frac{\int_{1/(\nu+1)}^{1}\alpha_2(\tau)d\tau}{A(\nu)}=\frac{b_2(\nu)}{A(\nu)},$$

then

$$P_2\le C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu,\nu)(\mu\nu)^{r\left(1-\frac{2}{p}\right)}B_{\mu,\nu}^r\left[\frac{b_1(\mu)b_2(\nu)}{A(\mu,\nu)}\right]^{\frac{r}{p}}. \qquad 3.3$$

Applying the Lemma 2.7 and Lemma 2.5, respectively, to the quantites P_3 and P_4, in the analogous way as we did for quanties P_1 and P_2, we arrive to these estimates:

$$P_3\le C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu,\nu)(\mu\nu)^{r\left(1-\frac{2}{p}\right)}B_{\mu,\nu}^r\left[\frac{b_2(\mu)b_1(\nu)}{A(\mu,\nu)}\right]^{\frac{r}{p}} \qquad 3.4$$

and

$$P_4\le C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu,\nu)(\mu\nu)^{r\left(1-\frac{2}{p}\right)}B_{\mu,\nu}^r\left[\frac{b_2(\mu)b_2(\nu)}{A(\mu,\nu)}\right]^{\frac{r}{p}}. \qquad 3.5$$

Inserting estimates (3.2)-(3.5) to (3.1) we have

$$\left\{\left\|f\right\|_{\psi,p,r}^{(m,n)}\right\}^r \le C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}A(\mu,\nu)(\mu\nu)^{r\left(1-\frac{2}{p}\right)}B_{\mu,\nu}^r\Bigg\{\left[\frac{b_1(\mu)b_1(\nu)}{A(\mu,\nu)}\right]^{\frac{r}{p}}$$

$$+\left[\frac{b_1(\mu)b_2(\nu)}{A(\mu,\nu)}\right]^{\frac{r}{p}}+\left[\frac{b_2(\mu)b_1(\nu)}{A(\mu,\nu)}\right]^{\frac{r}{p}}+\left[\frac{b_2(\mu)b_2(\nu)}{A(\mu,\nu)}\right]^{\frac{r}{p}}\Bigg\}. \qquad 3.6$$

Owing to the Lemma 2.8, relation (3.6) takes the following form

$$\left\{\| f \|_{\psi,p,r}^{(m,n)}\right\}^r \le C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} b(\mu,\nu)(\mu\nu)^{r\left(1-\frac{2}{p}\right)} B_{\mu,\nu}^r \left[\frac{b(\mu,\nu)}{A(\mu,\nu)}\right]^{\frac{r}{p}-1}.$$

Let us consider the case when $r<p$. We start with estimate of $P_i, i=1,2,3,4$, alternately.

Estimating of P_1: Using two dimensional case of Lemma 2.9, which can be prove simply applying it twice, and changing the order of summation we have

$$P_1 \le C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} A(\mu)A(\nu)\mu^{-mr}\nu^{-nr}\sum_{i=1}^{\mu}\sum_{k=1}^{\nu} i^{(m+1)r-\frac{2r}{p}} k^{(n+1)r-\frac{2r}{p}} B_{i,k}^r$$
$$= C\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} i^{(m+1)r-\frac{2r}{p}} k^{(n+1)r-\frac{2r}{p}} B_{i,k}^r \sum_{\mu=i}^{\infty}\sum_{\nu=k}^{\infty} A(\mu)A(\nu)\mu^{-mr}\nu^{-nr}.$$

Is not dificult to prove estimates: $i^{mr}\sum_{\mu=i}^{\infty}\frac{A(\mu)}{\mu^{mr}} \le 2^{mr} b_1(i)$ and

$k^{nr}\sum_{\nu=k}^{\infty}\frac{A(\nu)}{\nu^{nr}} \le 2^{nr} b_1(k)$, therefore

$$P_1 \le C\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(ik)^{r\left(1-\frac{2}{p}\right)} B_{i,k}^r b_1(i)b_1(k). \quad 3.7$$

Estimating of P_2: Again, applying two dimensional case of Lemma 2.9 and changing the order of summation we get

$$P_2 \le C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} A(\mu)A(\nu)\mu^{-mr}\sum_{i=1}^{\mu}\sum_{k=\nu}^{\infty} i^{(m+1)r-\frac{2r}{p}} k^{r-\frac{2r}{p}} B_{i,k}^r$$
$$= C\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} i^{(m+1)r-\frac{2r}{p}} k^{r-\frac{2r}{p}} B_{i,k}^r \sum_{\mu=i}^{\infty}\sum_{\nu=1}^{k} A(\mu)A(\nu)\mu^{-mr}$$
$$\le C\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(ik)^{r\left(1-\frac{2}{p}\right)} B_{i,k}^r b_1(i)b_2(k), \quad 3.8$$

because $i^{mr}\sum_{\mu=i}^{\infty}\frac{A(\mu)}{\mu^{mr}} \le 2^{mr} b_1(i)$ and $\sum_{\nu=1}^{k} A(\nu) = b_2(k)$.

Estimating of P_3 *and* P_4: In the same manner we get these estimates:

$$P_3 \le C\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(ik)^{r\left(1-\frac{2}{p}\right)} B_{i,k}^r b_2(i)b_1(k). \quad 3.9$$

and

$$P_4 \le C\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(ik)^{r\left(1-\frac{2}{p}\right)} B_{i,k}^r b_2(i)b_2(k). \quad 3.10$$

Estimating of (3.1) together with estimates (3.7), (3.8), (3.9) and (3.10) give

$$\left\{\| f \|_{\psi,p,r}^{(m,n)}\right\}^r \le C\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(ik)^{r\left(1-\frac{2}{p}\right)} B_{i,k}^r b(i,k),$$

which fully demonstrates Theorem 3.1.

Note that these inequalities in our result and all forthcoming ones are understood so that the finiteness of the right-hand side implies the finiteness of the left-hand side. The following theorem gives sufficient conditions in terms of Fourier coefficients so that a function belongs to the classes $\Lambda^{m,n}(\psi,p,r)$.

Theorem: 3.2. Let m,n be natural numbers, $1<p\le 2$, $1\le r<\infty$ and $1/p+1/q=1$. If $a_{j,s}$ are coefficients either of the series (1.1), (1.2), (1.3) or (1.4), then:

(1) For $r\le q$ we have

$$C\left\{\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} b_1(\mu)b_1(\nu)|a_{\mu,\nu}|^r\left[\frac{b_1(\mu)b_1(\nu)}{A(\mu,\nu)}\right]^{\frac{r}{q}-1}\right\}^{\frac{1}{r}} \le \| f \|_{\psi,p,r}^{(m,n)};$$

(2) For $r>q$ we have

$$C\left\{\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} b_1(\mu)b_1(\nu)|a_{\mu,\nu}|^r\right\}^{\frac{1}{r}} \le \| f \|_{\psi,p,r}^{(m,n)}.$$

Proof: Let $f(x,y)$ be even function with respect to both variables (three other cases can be consider in a similar manner). Then is not difficult to show that Fourier series of $\Delta_{m,n} f(x,y;t,\tau)$ is as follows

$$\begin{cases} (-1)^{\frac{m+n}{2}} 2^{m+n}\sum_{j=1}^{\infty}\sum_{s=1}^{\infty} a_{j,s}\cos jx\cos sy\sin^m jt\sin^n s\tau \text{ if } m,n \text{ are even} \\ (-1)^{\frac{m+n-1}{2}} 2^{m+n}\sum_{j=1}^{\infty}\sum_{s=1}^{\infty} a_{j,s}\sin jx\cos sy\sin^m jt\sin^n s\tau \text{ if } m \text{ even and } n \text{ odd} \\ (-1)^{\frac{m+n-1}{2}} 2^{m+n}\sum_{j=1}^{\infty}\sum_{s=1}^{\infty} a_{j,s}\cos jx\sin sy\sin^m jt\sin^n s\tau \text{ if } m \text{ odd and } n \text{ even} \\ (-1)^{\frac{m+n-2}{2}} 2^{m+n}\sum_{j=1}^{\infty}\sum_{s=1}^{\infty} a_{j,s}\sin jx\sin sy\sin^m jt\sin^n s\tau \text{ if } m, n \text{ are odd.} \end{cases}$$

By the well-known Hausdorf-Young's theorem we have

$$C\left(\int_0^{2\pi}\int_0^{2\pi}\left|\Delta_{m,n}f(x,y;t,\tau)\right|^p dxdy\right)^{\frac{r}{p}} \geq \left(\sum_{j=1}^{\infty}\sum_{s=1}^{\infty}\left|a_{j,s}\right|^q\left|\sin jt\right|^{mq}\left|\sin s\tau\right|^{nq}\right)^{\frac{r}{q}}.$$

Therefore,

$$\left\{\left\|f\right\|_{\psi,p,r}^{(m,n)}\right\}^r \geq C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}\int_{1/(\mu+1)}^{1/\mu}\int_{1/(\nu+1)}^{1/\nu}\alpha_1(t)\alpha_2(\tau)\left(\sum_{j=1}^{\mu}\sum_{s=1}^{\nu}\left|a_{j,s}\right|^q\left|\sin jt\right|^{mq}\left|\sin s\tau\right|^{nq}\right)^{\frac{r}{q}}dtd\tau.$$

Applying twice the inequality $\sin u \geq \dfrac{2}{\pi}u$ for $0 \leq u \leq \pi/2$, then the following estimate holds

$$\left\{\left\|f\right\|_{\psi,p,r}^{(m,n)}\right\}^r \geq C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}\mu^{-mr}\nu^{-nr}A(\mu)A(\nu)\left(\sum_{j=1}^{\mu}\sum_{s=1}^{\nu}\left|a_{j,s}\right|^q j^{mq}s^{nq}\right)^{\frac{r}{q}}.$$

Let $r \leq q$, then by the Lemma 2,4 we have

$$\left\{\left\|f\right\|_{\psi,p,r}^{(m,n)}\right\}^r \geq C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}\mu^{-mr}\nu^{-nr}A(\mu)A(\nu)\left(\left|a_{\mu,\nu}\right|^q\mu^{mq}\nu^{nq}\beta_\mu^{(1)}\beta_\nu^{(2)}\right)^{\frac{r}{q}}.$$

It is easy to prove the following estimates $\beta_\mu^{(1)} \geq b_1(\mu)/A(\mu)$ and $\beta_\nu^{(2)} \geq b_1(\nu)/A(\nu)$, therefore the last estimate takes the form

$$\left\{\left\|f\right\|_{\psi,p,r}^{(m,n)}\right\}^r \geq C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}b_1(\mu)b_1(\nu)\left|a_{\mu,\nu}\right|^r\left[\frac{b_1(\mu)b_1(\nu)}{A(\mu,\nu)}\right]^{\frac{r}{q}-1}. \tag{3.11}$$

Let $r > q$, then using the two dimensional case of the Lemma 2.9 and changing the order of the summation we have

$$\begin{aligned}\left\{\left\|f\right\|_{\psi,p,r}^{(m,n)}\right\}^r &\geq C\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}\mu^{-mr}\nu^{-nr}A(\mu)A(\nu)\sum_{j=1}^{\mu}\sum_{s=1}^{\nu}\left|a_{j,s}\right|^r j^{mr}s^{nr}\\ &= C\sum_{j=1}^{\infty}\sum_{s=1}^{\infty}\left|a_{j,s}\right|^r j^{mr}s^{nr}\sum_{\mu=j}^{\infty}\sum_{\nu=s}^{\infty}\mu^{-mr}\nu^{-nr}A(\mu)A(\nu)\\ &\geq C\sum_{j=1}^{\infty}\sum_{s=1}^{\infty}\left|a_{j,s}\right|^r b_1(j)b_1(s).\end{aligned} \tag{3.12}$$

Estimates (3.11) and (3.12) prove Theorem 2.2.

4. COROLLARIES:

Here, in the following, we are going to deduce some corollaries from Theorem 3.1 and Theorem 3.2. We noticed in the section 3 that there existed two conditions, separately, in each proved theorem which provide that a function in two variables belongs new classes considered in this paper. Therefore arises the question: Does exist only a single condition in Theorem 3.1 that includes both its conditions? Under some additional conditions the answer is positive for both theorems.

Corollary: 4.1. Under the conditions of Theorem 3.1 and with $b(\mu,\nu) \leq CA(\mu,\nu)$ we have:

$$\left\|f\right\|_{\psi,p,r}^{(m,n)} \leq C\left\{\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}b(\mu,\nu)(\mu\nu)^{r\left(1-\frac{2}{p}\right)}B_{\mu,\nu}^r\right\}^{\frac{1}{r}}.$$

Corollary: 4.2. Under the conditions of Theorem 3.2 and with $b_1(\mu)b_1)\nu) \geq CA(\mu,\nu)$ we have:

$$C\left\{\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}b_1(\mu)b_1(\nu)\,|\,a_{\mu,\nu}\,|^r\right\}^{\frac{1}{r}} \leq \left\|f\right\|_{\psi,p,r}^{(m,n)}.$$

Corollary: 4.3. Under the conditions of Corollary 4.1 and if Fourier coefficients $a_{j,s}$ are monotone in the sense of Hardy, see [1], then:

$$\left\|f\right\|_{\psi,p,r}^{(m,n)} \leq C\left\{\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}b(\mu,\nu)(\mu\nu)^{r\left(1-\frac{2}{p}\right)}a_{\mu,\nu}^r\right\}^{\frac{1}{r}}.$$

As a special case, for $\alpha_1(t) = t^{-\alpha r-1}$ and $\alpha_2(\tau) = \tau^{-\beta r-1}$, $(\alpha,\beta > 0)$, it is easy to prove the following estimates:

$A(\mu,\nu) \leq C\mu^{\alpha r-1}\nu^{\beta r-1}$ and $b(\mu,\nu) \leq C\mu^{\alpha r}\nu^{\beta r}$.

Therefore, from last estimates and the Corollaries 4.1 and 4.2 respectively, the following corollaries hold.

Corollary: 4.4.[6] Let

$0 < \alpha \leq m, 0 < \beta \leq n, 1 < p < \infty, 1 \leq r < \infty, 1/p + 1/q = 1$

and $a_{\mu,\nu}$ are the coefficients of the series (1.1). If

$$\sum_{j=1}^{\infty}\sum_{s=1}^{\infty}\left|\Delta_{1,1}a_{j,s}\right| < \infty,$$

then

$$\left\|f\right\|_{\alpha,\beta,p,r}^{(m,n)} \leq C\left\{\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}\mu^{r\left(\alpha+\frac{1}{q}\right)-1}\nu^{r\left(\beta+\frac{1}{q}\right)-1}\left(\sum_{j=\mu}^{\infty}\sum_{s=\nu}^{\infty}\left|\Delta_{1,1}a_{j,s}\right|\right)^r\right\}^{\frac{1}{r}}.$$

Corollary: 4.5. [6] Let

$0 < \alpha \leq m, 0 < \beta \leq n, 1 < p < \infty, 1 \leq r < \infty, 1/p + 1/q = 1$

and $a_{\mu,\nu}$ are the coefficients of the series (1.1) monotone in sense of Hardy. If

$$\left(\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}\mu^{r\left(\alpha+\frac{1}{q}\right)-1}\nu^{r\left(\beta+\frac{1}{q}\right)-1}a_{\mu,\nu}^{r}\right)^{\frac{1}{r}}<\infty,$$

then $f \in \Lambda^{m,n}(\alpha, \beta, p, r)$.

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